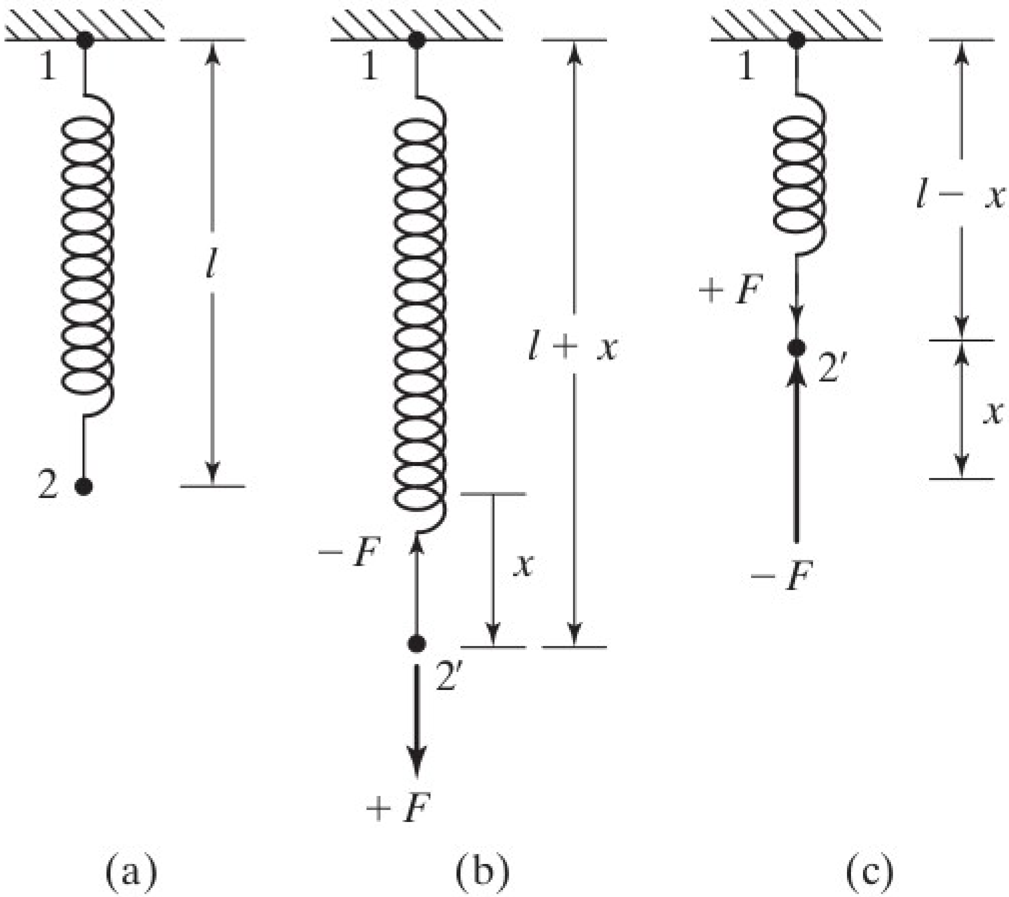
Mechanical Vibrations

Prof. Paulo J. Paupitz Gonçalves

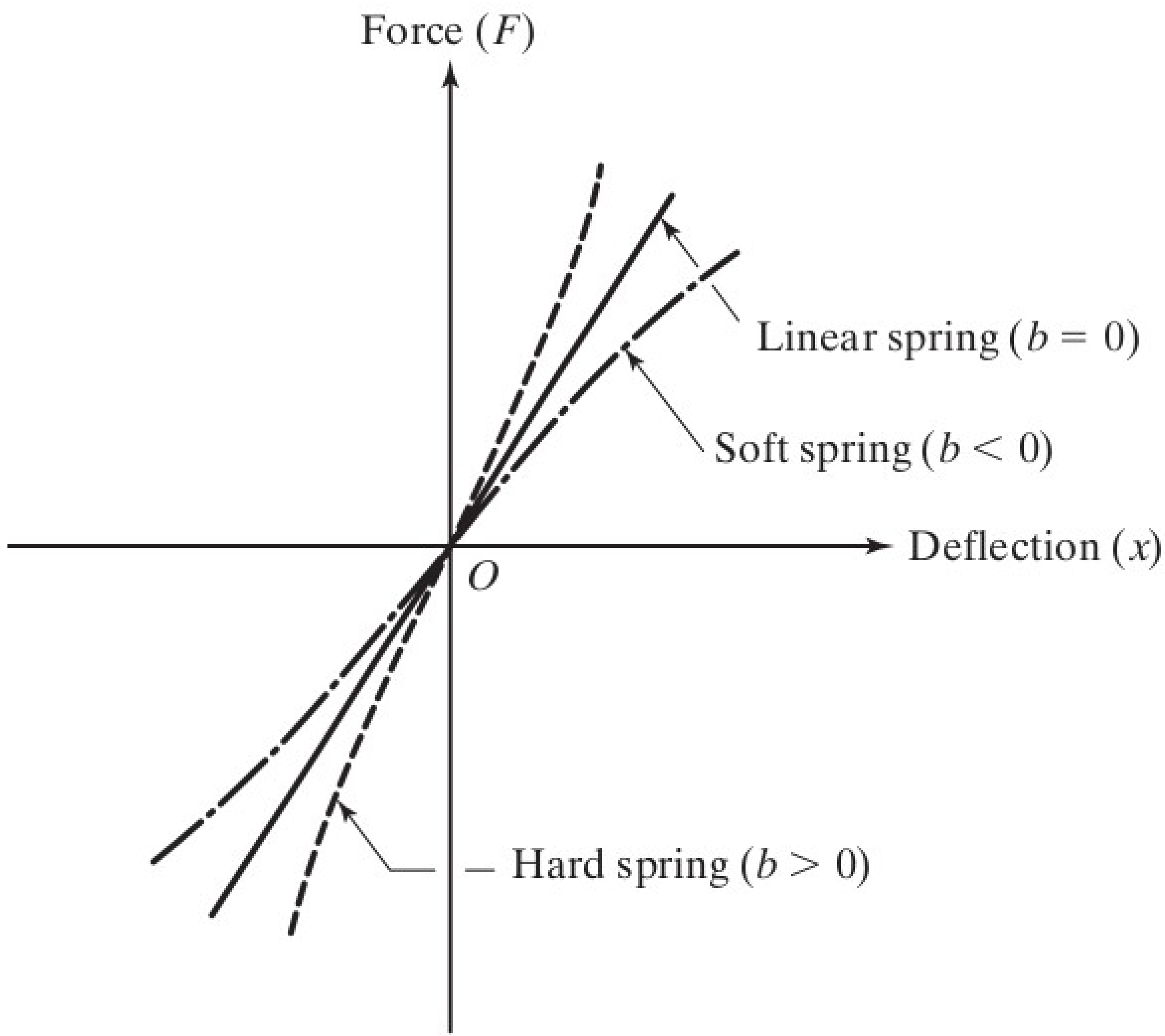
More on Stiffness (Spring Elements)



Work Done

$$U = \frac{1}{2}kx^2$$

Nonlinear Springs



Restoring Force

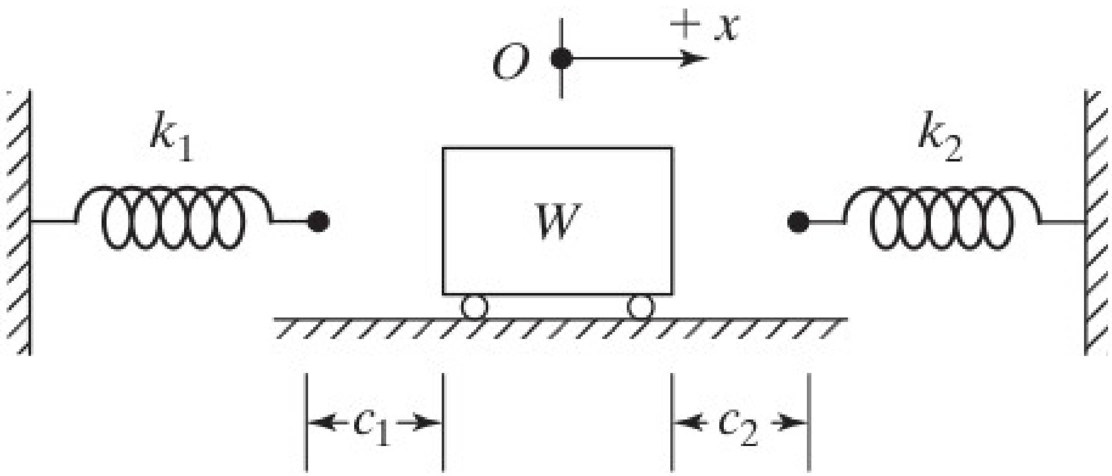
$$F = ax + bx^3$$

Examples of Springs

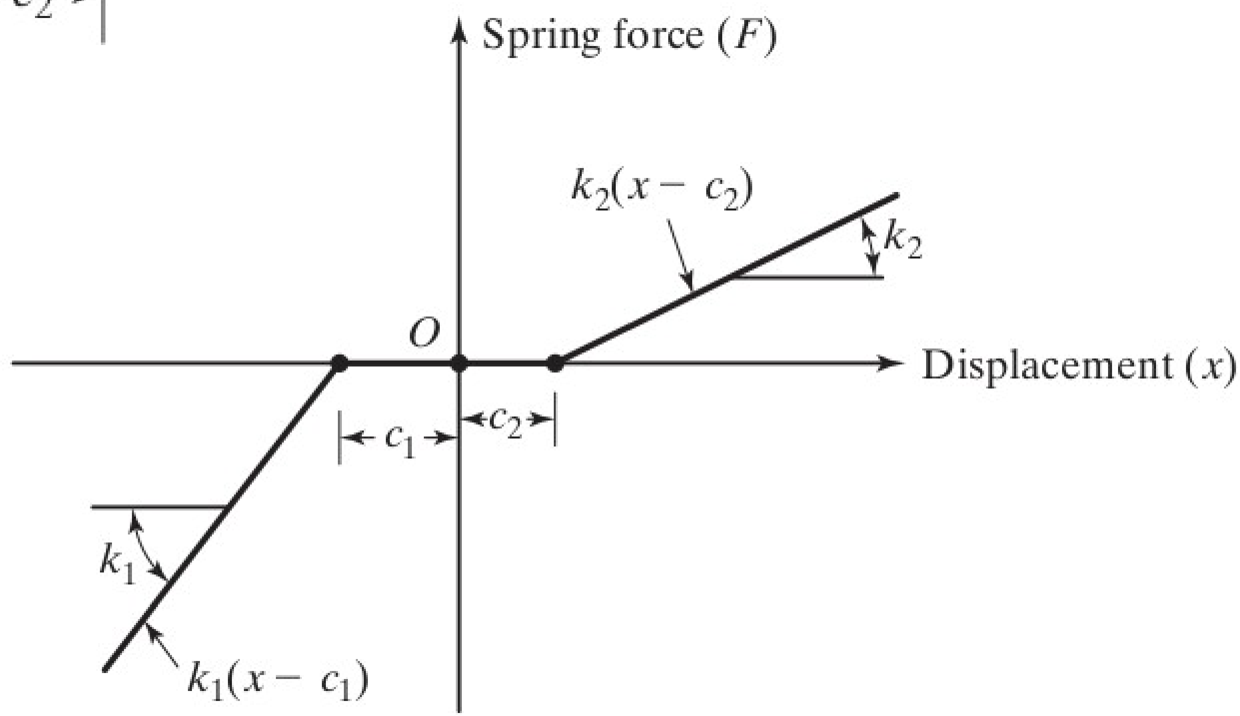





Nonlinear Stiffness

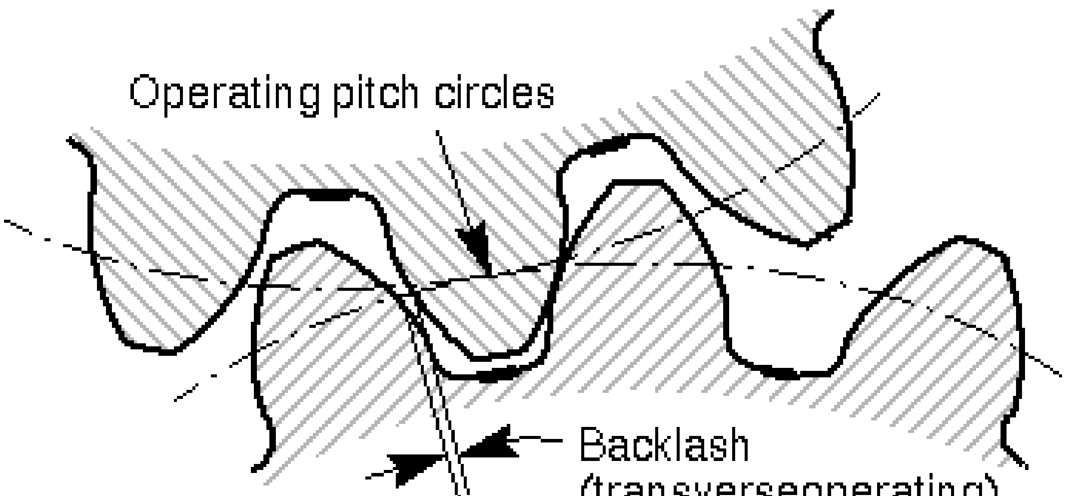


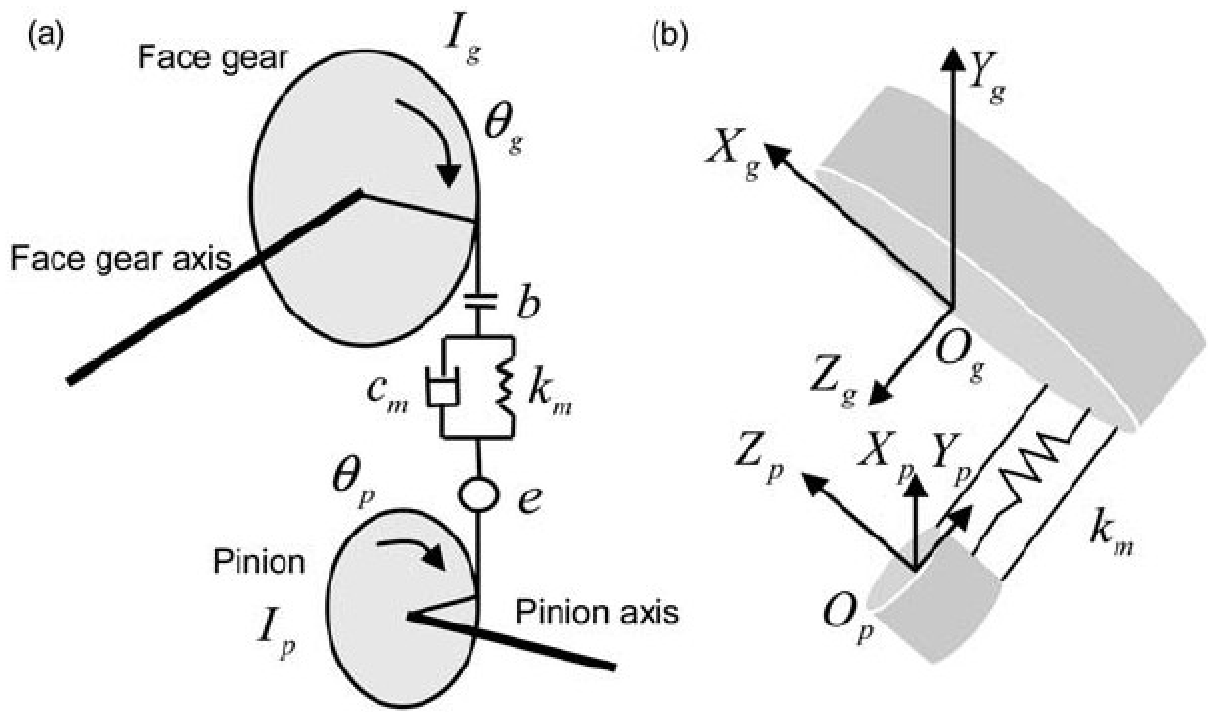
- Freeplay
- Dead Zone
- Backlash



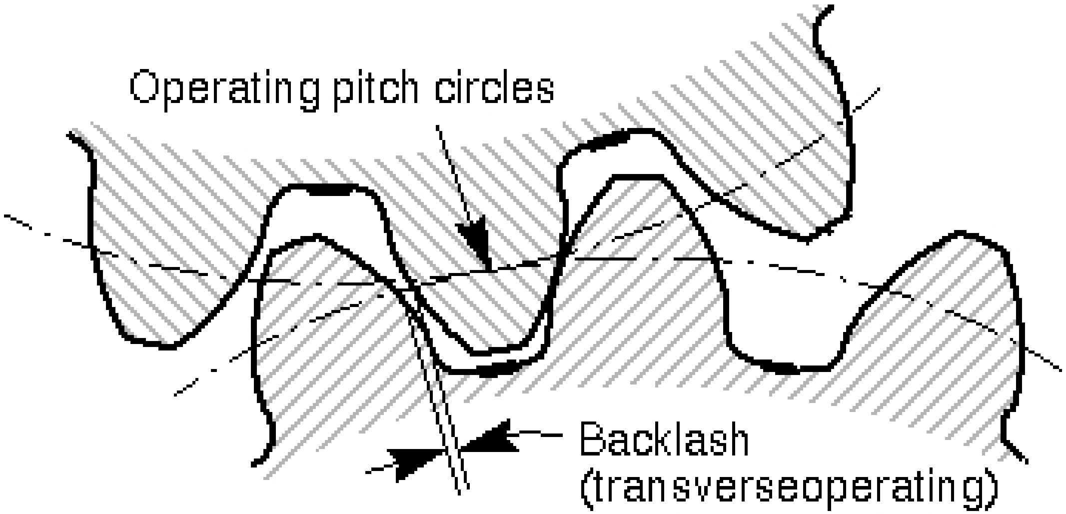
(b)

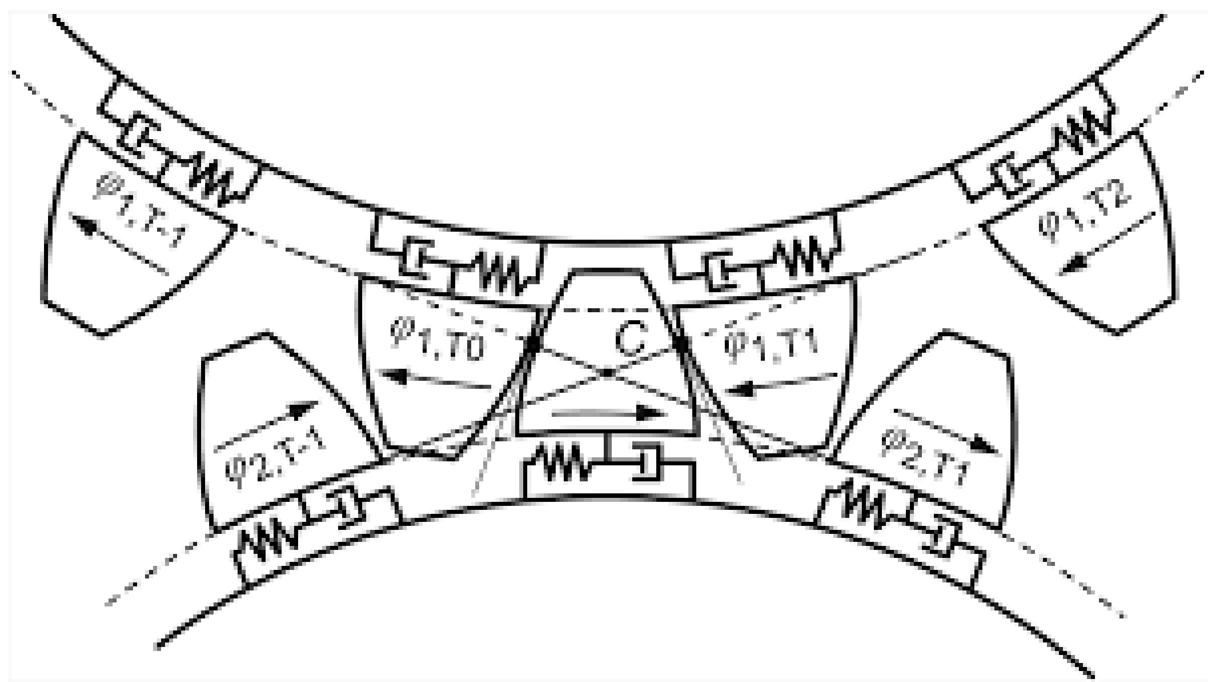
Gear Backlash Vibration Models





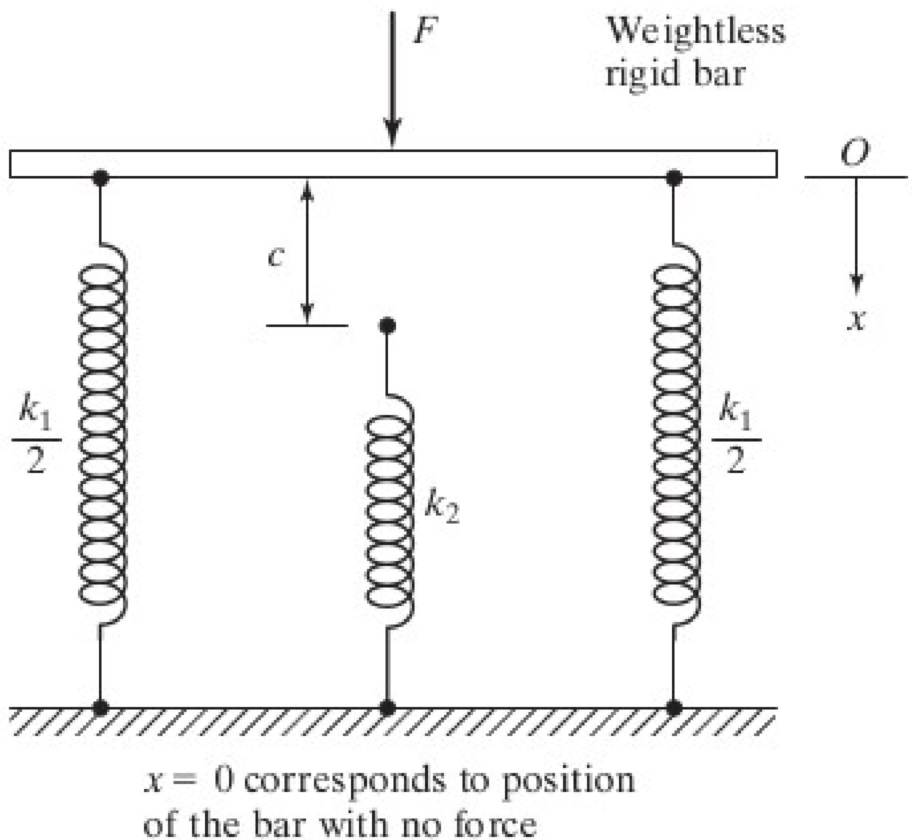
Gear Backlash Vibration Models

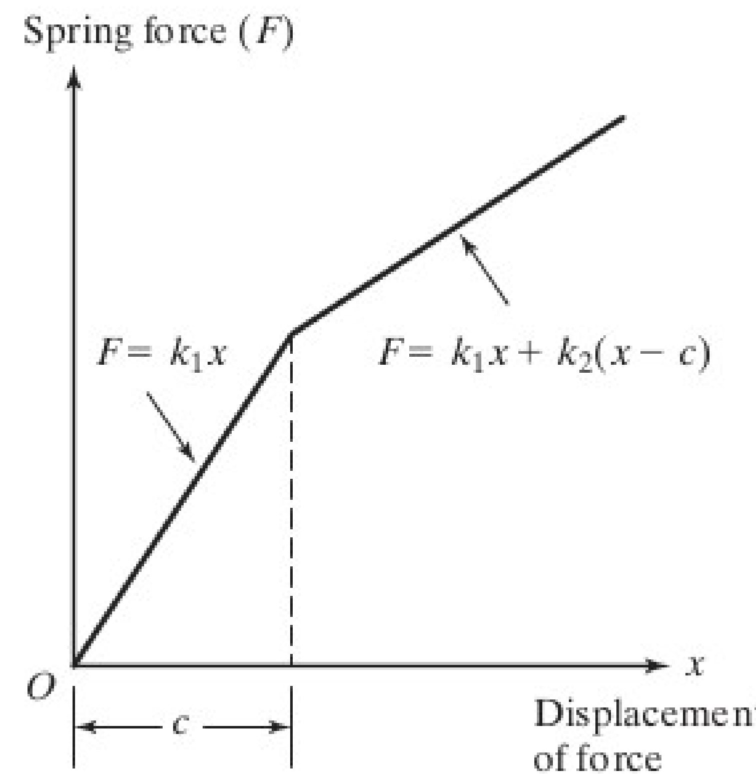




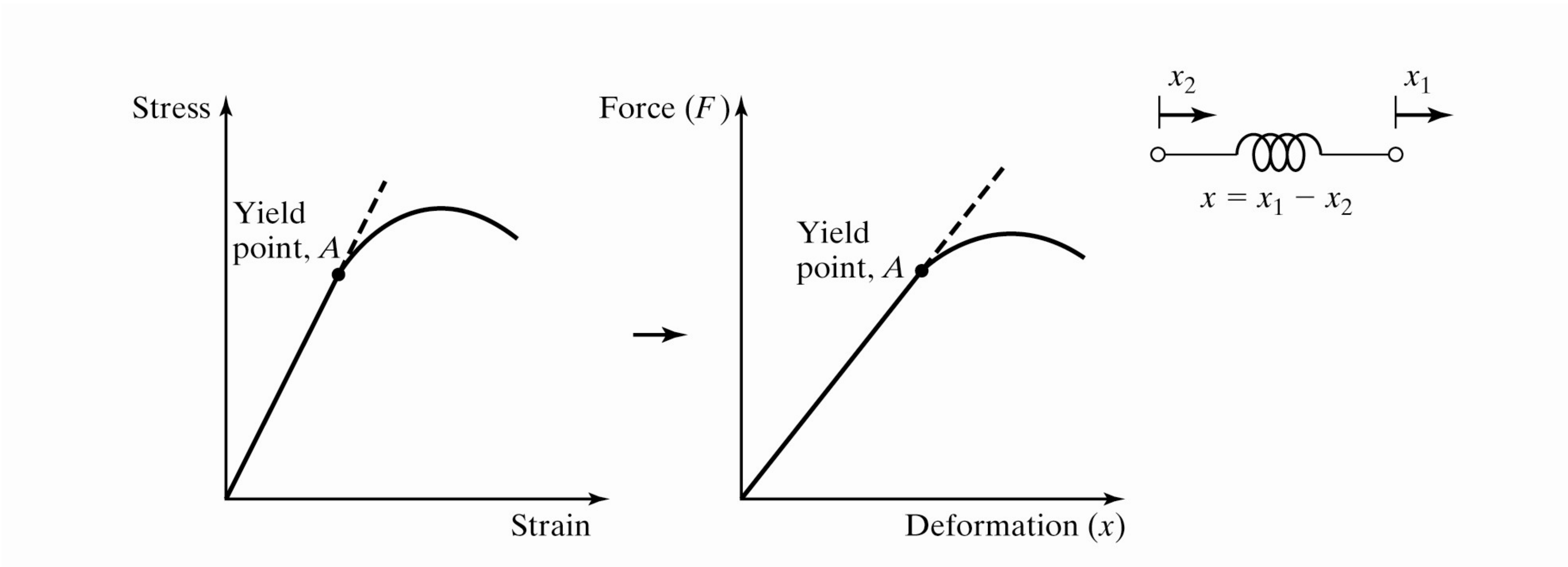
Nonlinear Stiffness

Bilinear Stiffness



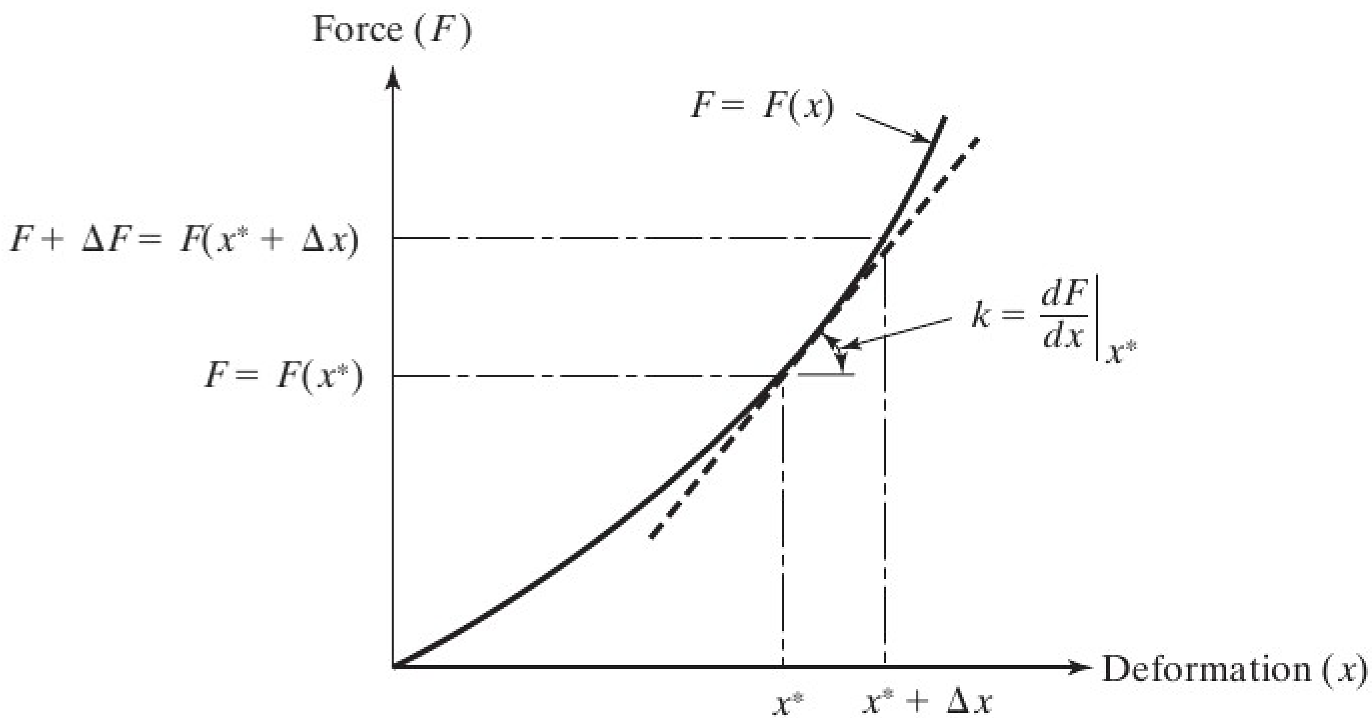


More on Stiffness (Spring Elements)



Often springs are nonlinear. The behavior shown in the figure (force as a function of the deformation) exhibits nonlinear behavior after point A.

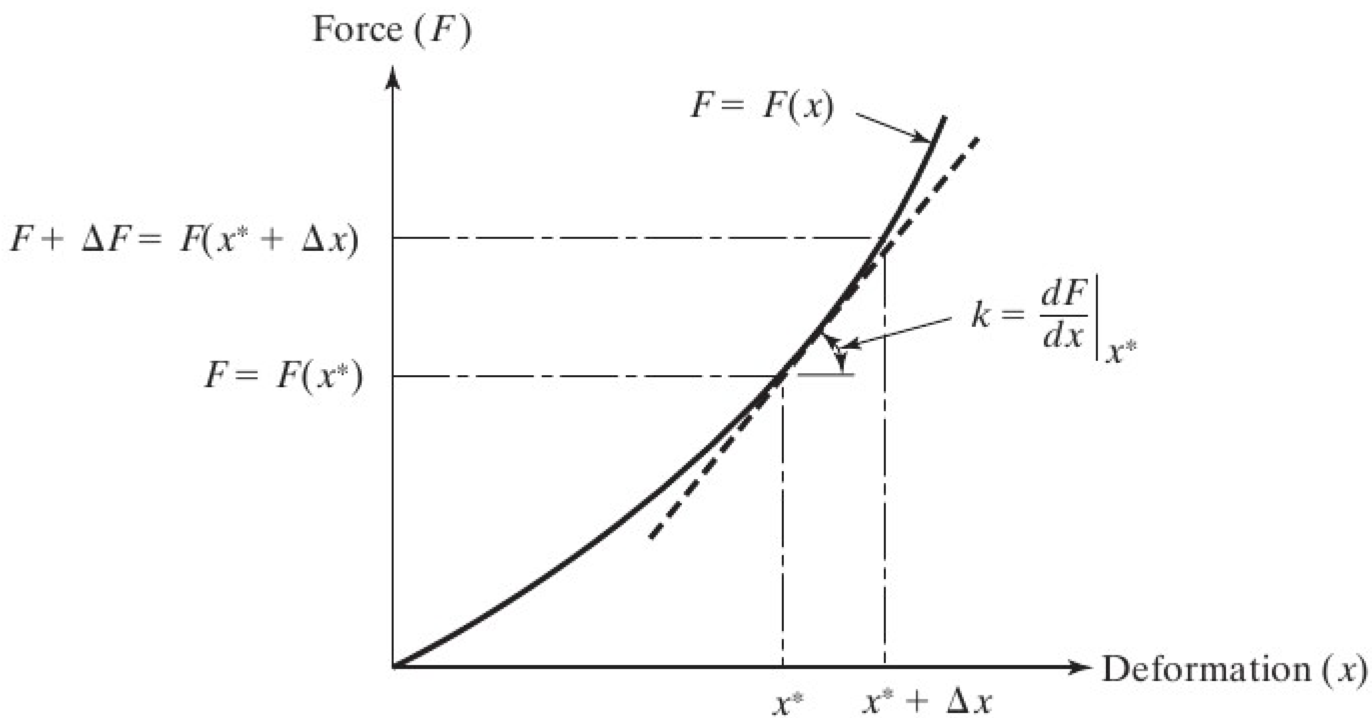
Linearisation of a Nonlinear Spring



$$F + \Delta F = F\left(x^* + \Delta x\right)$$

$$F + \Delta F = F\left(x^* + \Delta x\right) = F(x^*) + \frac{dF}{dx}\bigg|_{x^*}(\Delta x) + \frac{1}{2!}\frac{d^2F}{dx^2}\bigg|_{x^*}(\Delta x)^2 + \cdots$$

Linearisation of a Nonlinear Spring



For small displacements around an equilibrium point, the first derivative is usually enough to represent the stiffness

$$F + \Delta F = F\left(x^* + \Delta x\right) = F(x^*) + \left.\frac{dF}{dx}\right|_{x^*} (\Delta x)$$

Linearisation of a Nonlinear Spring

Since $F = F(x^*)$

It is possible to express ΔF as

$$\Delta F = k \Delta x$$

where k is the linearised stiffness constant at x^*

$$k = \left. \frac{dF}{dx} \right|_{x^*}$$

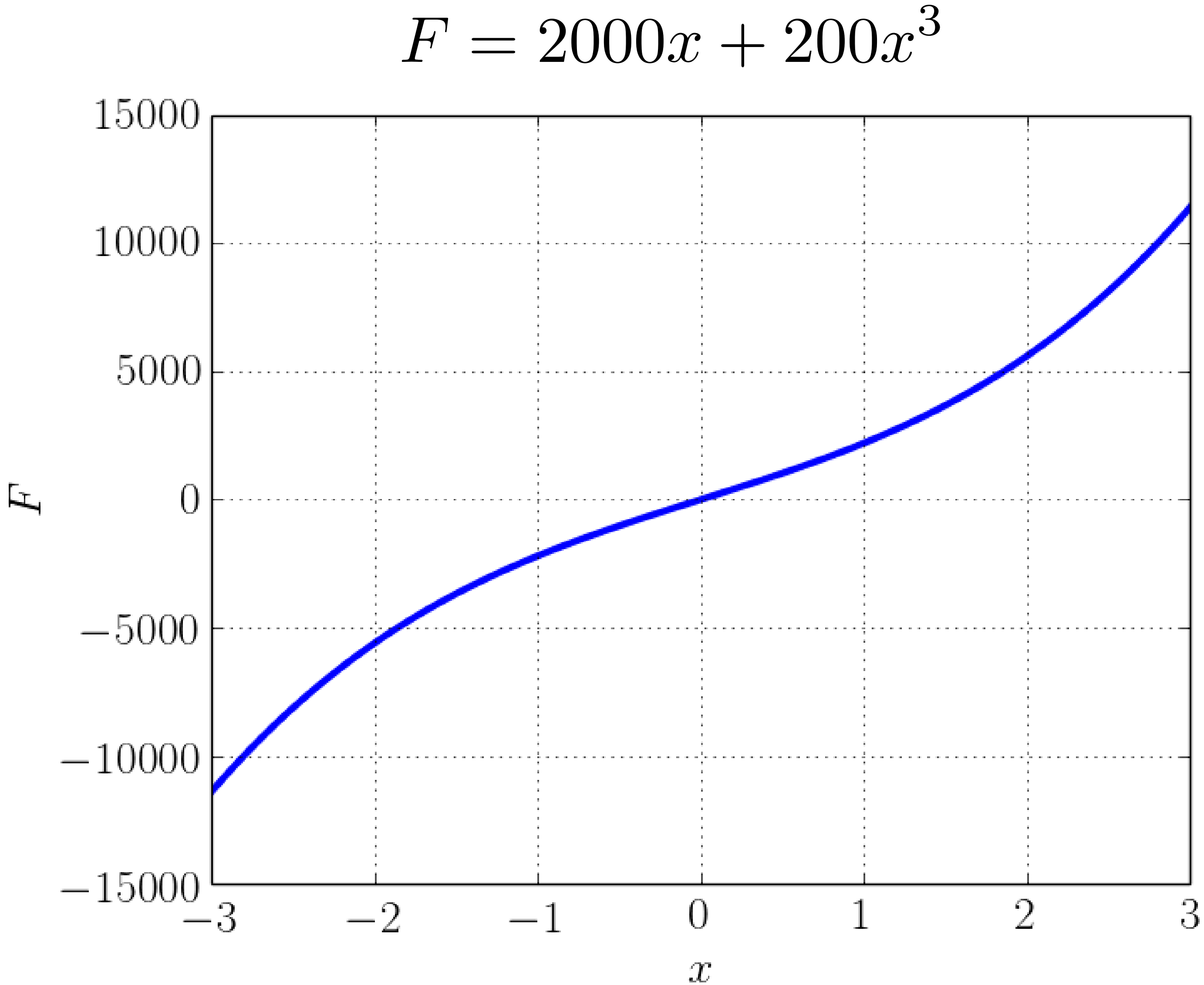
Example

A precision milling machine, weighing 1000 lb, is supported on a rubber mount. The force-deflection relationship of the rubber mount is given by

$$F = 2000x + 200x^3$$

where the force (F) and the deflection (x) are measured in pounds and inches, respectively. Determine the equivalent linearised spring constant of the rubber mount at its static equilibrium position.

Example (Solution)



Example (Solution)

Solution: The static equilibrium position of the rubber mount (x^*), under the weight of the milling machine, can be determined from

$$1000 = 2000(x^*) + 200(x^*)^3$$

or

$$200(x^*)^3 + 2000(x^*) - 1000 = 0$$

The roots of this equation can be calculate using for instance the command roots in Matlab or Octave

Example (Solution)

```
>> P = [200 0 2000 -1000]
>> roots(P)
ans =

   0.24418 + 3.19043i
   0.24418 - 3.19043i
  -0.48835 + 0.00000i
```

We use the only real root to obtain the linearised stiffness

Example (Solution)

Such that

$$k = \left.\frac{dF}{dx}\right|_{x^*}$$

and

$$F = 2000x + 200x^3$$

therefore

$$k = 2000 + 600(x^*)^2$$

$$k = 2000 + 600(0.4884)^2 = 2143.1207 \text{ lb/in}$$

Example (Solution)

And the static equilibrium position can be obtained

$$x = \frac{F}{k} = \frac{1000}{2143.1207} = 0.4666 \ \text{in}$$

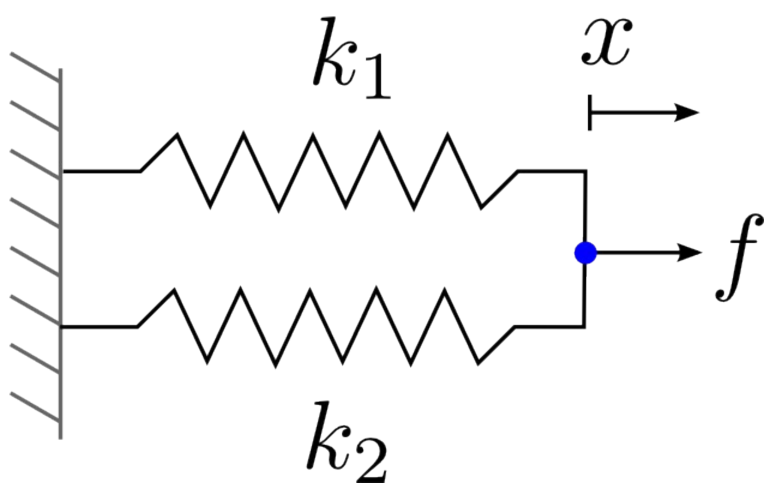
Problem

The force-deflection relation of a steel helical spring used in an engine is found experimentally as

$$F(x) = 200x + 50x^2 + 10x^3$$

where the force (F) and deflection (x) are measured in pounds and inches, respectively. If the spring undergoes a steady deflection of 0.5 in. during the operation of the engine, determine the equivalent linear spring constant of the spring at its steady deflection.

More on Stiffness (Spring Elements)



Parallel Combination

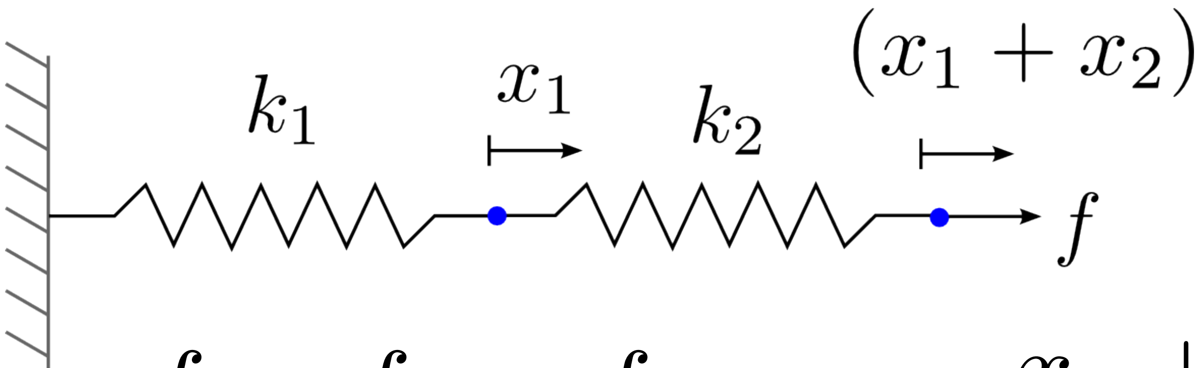
$$f = k_{\text{eq}} x$$

$$f = f_1 + f_2 = k_1 x + k_2 x$$

$$(k_1 + k_2)x = k_{\text{eq}} x$$

$$k_{\text{eq}} = (k_1 + k_2)$$

More on Stiffness (Spring Elements)



$$f = f_1 = f_2 \qquad x_1 + x_2 = x$$

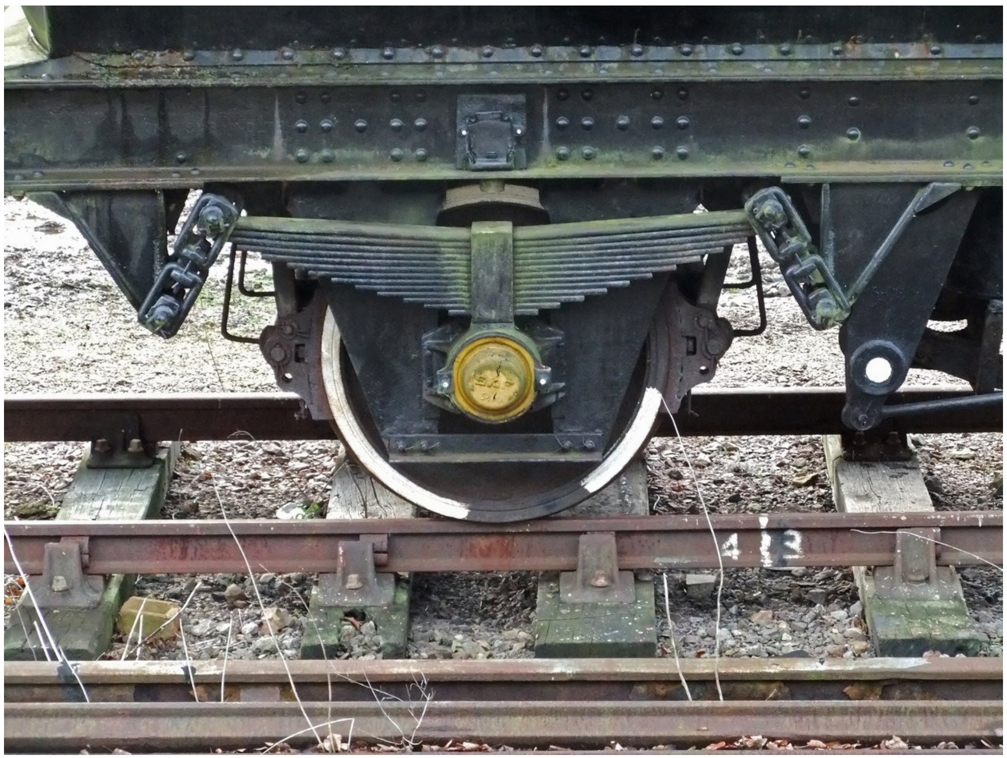
$$x_1 + x_2 = \frac{f}{k_1} + \frac{f}{k_2}$$

$$\frac{1}{k_{\text{eq}}} = \frac{1}{k_1} + \frac{1}{k_2}$$

Series Combination

Springs Combination Examples

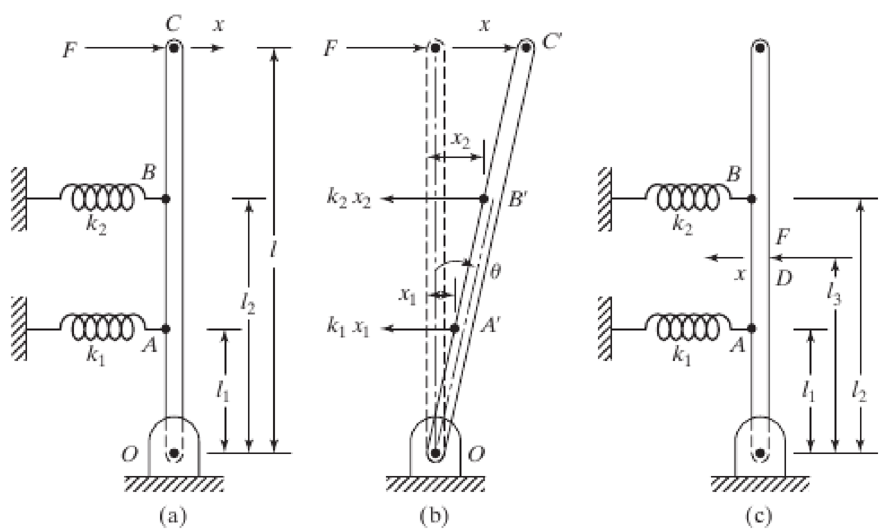
Springs Combination Examples



Springs Arrangements Examples

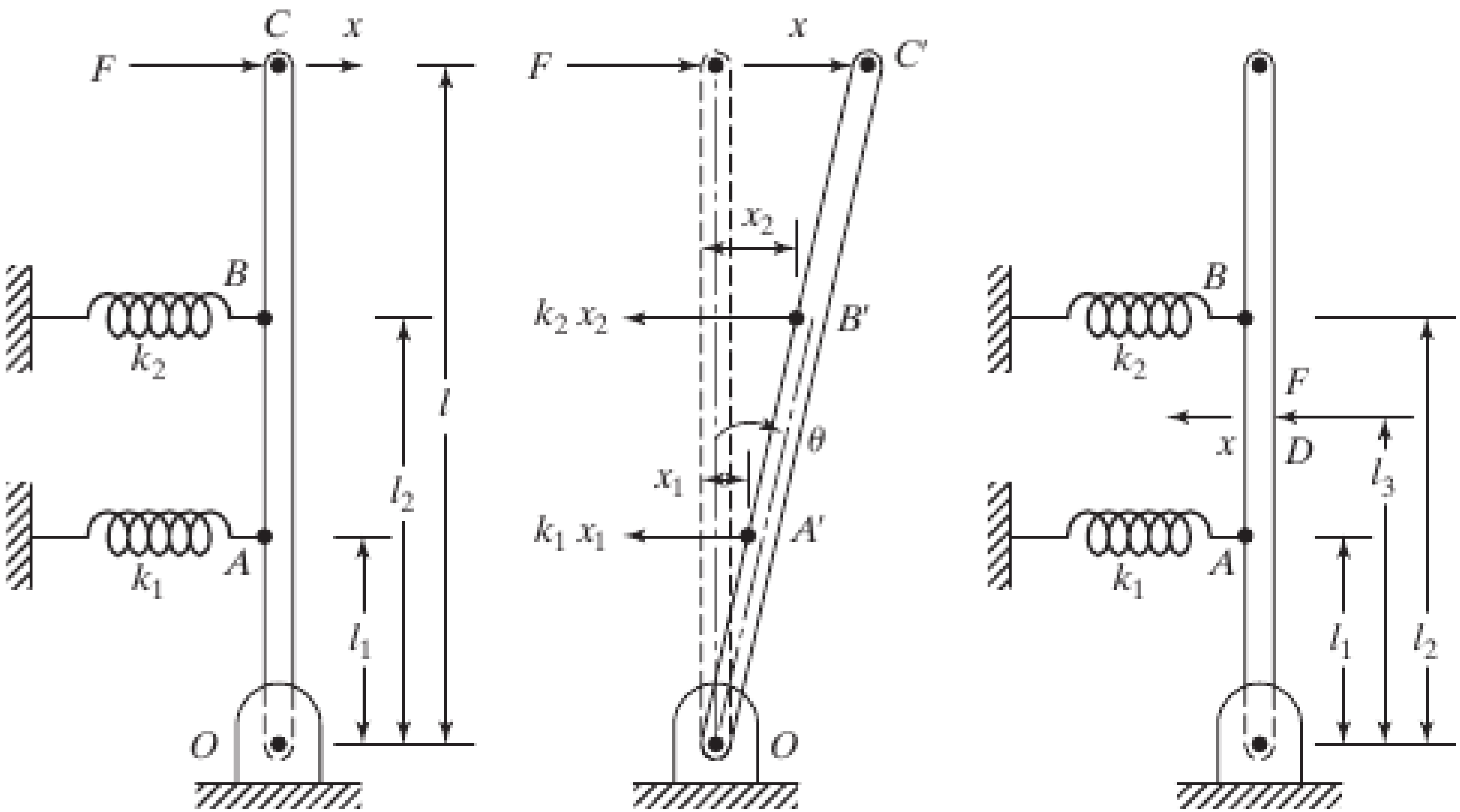
Example

A hinged rigid bar of length l is connected by two springs of stiffness k_1 and k_2 and is subjected to a force as shown in the figure. Assuming that the angular displacement of the bar θ is small, find the equivalent spring constant of the system that relates the applied force to the resulting displacement x .



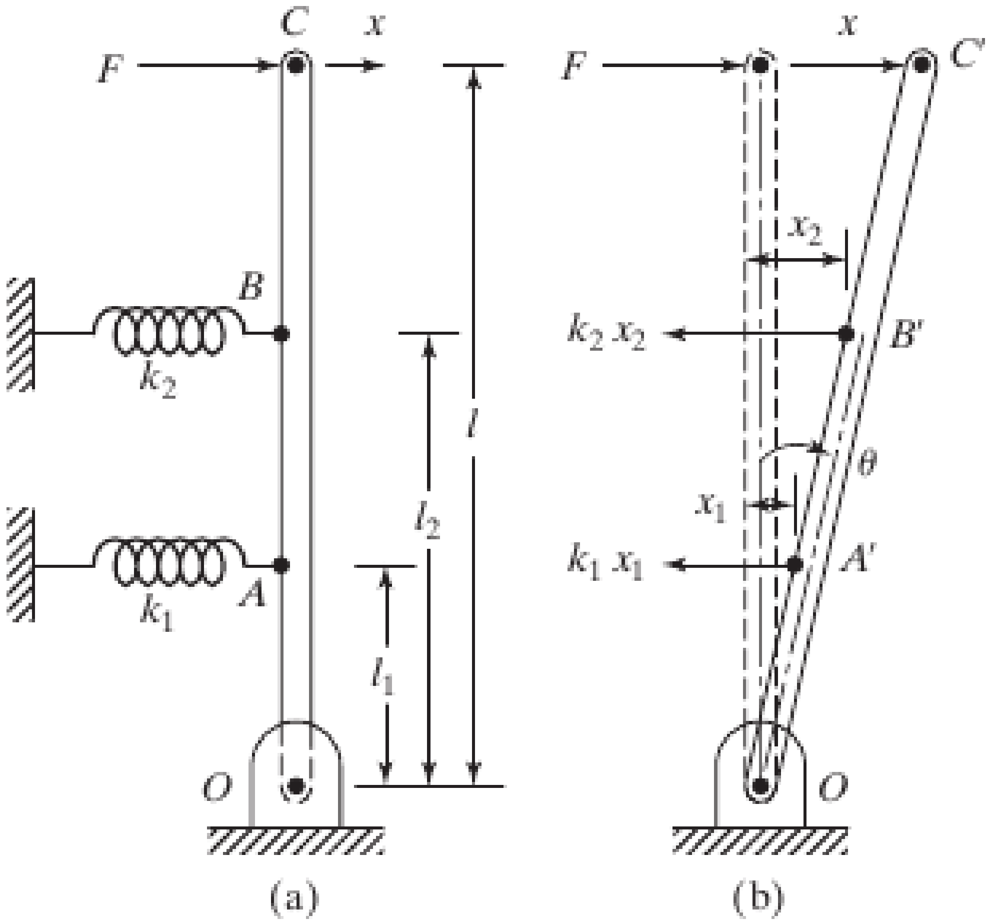
Example

Equivalent stiffness of a rigid bar connected by springs



(a) (b) (c)

Example (Solution)



$$x_1 = l_1 \sin\theta$$

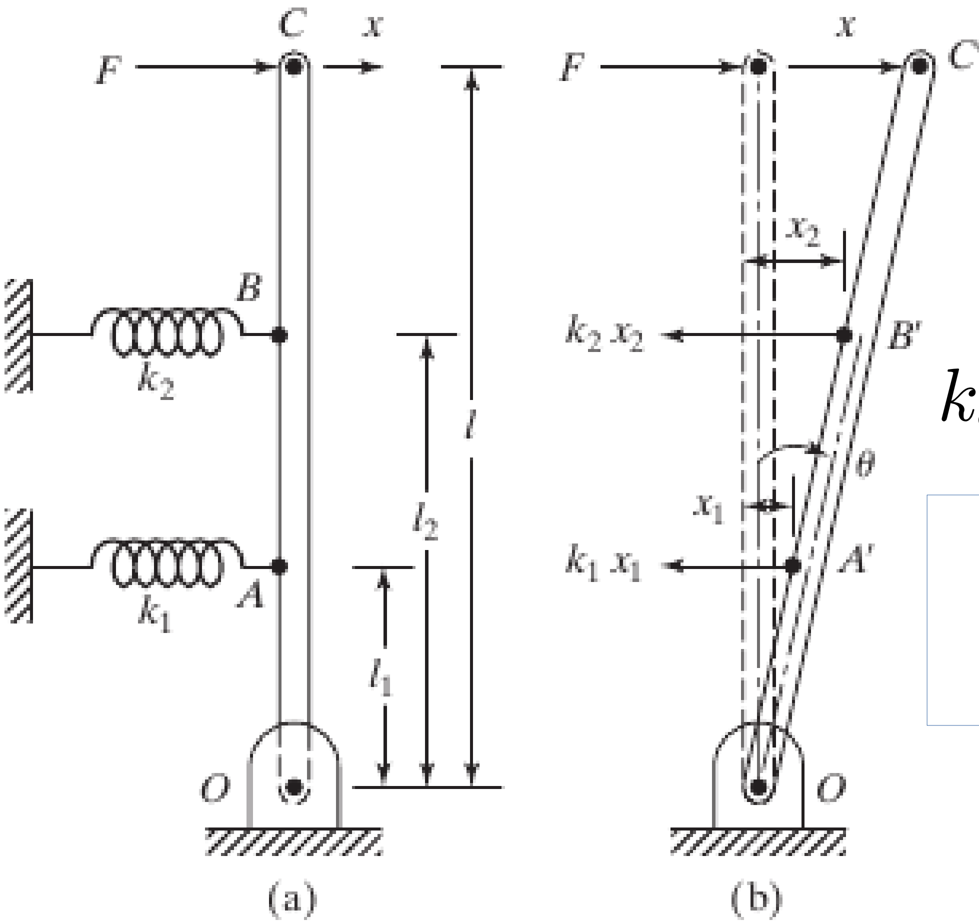
$$x_2 = l_2 \sin\theta$$

$$x = l \sin\theta$$

For small angles

$$\sin\theta \approx \theta$$

Example (Solution)



The moment equilibrium

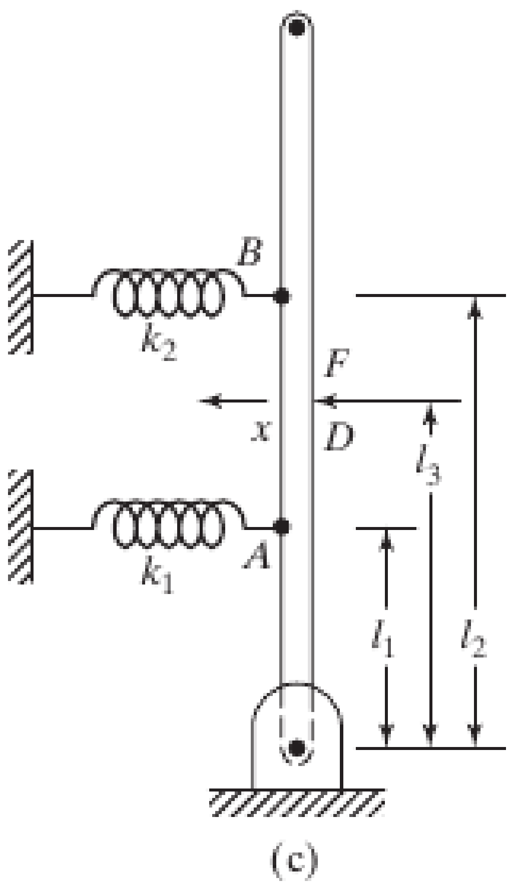
$$k_{\text{eq}}x \cdot (l) = k_1 x_1 \cdot (l_1) + k_2 x_2 \cdot (l_2)$$

$$x_1 = l_1 \sin\theta \qquad x_2 = l_2 \sin\theta$$

$$x = l \sin\theta$$

$$k_{\text{eq}} = k_1 \left(\frac{l_1}{l}\right)^2 + k_2 \left(\frac{l_2}{l}\right)^2$$

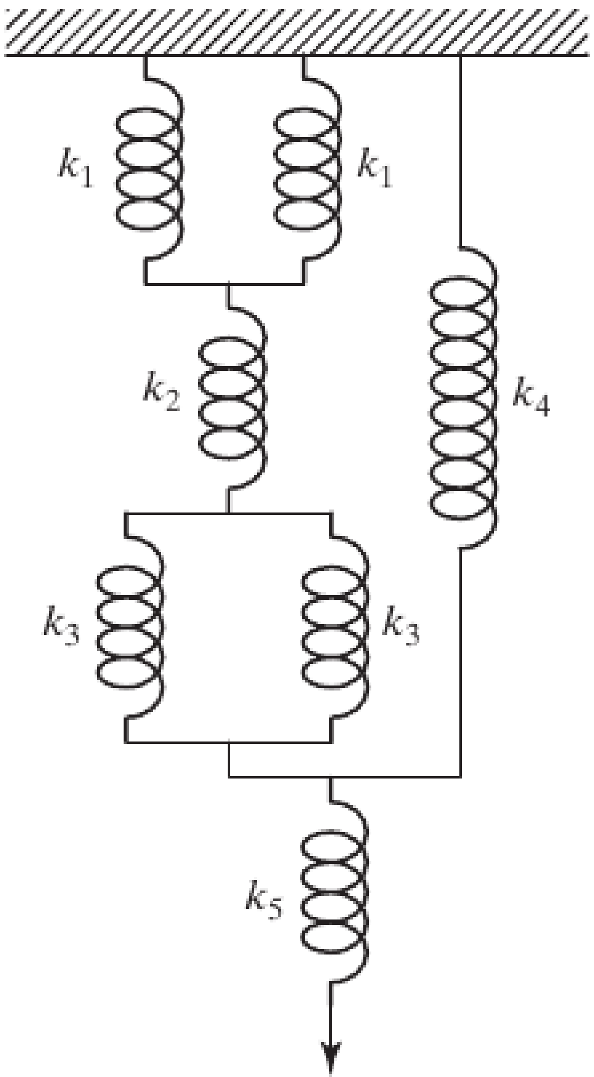
Example



If the force F is applied at another point D of the rigid bar as shown in the figure, the equivalent spring constant referred to point D can be found as

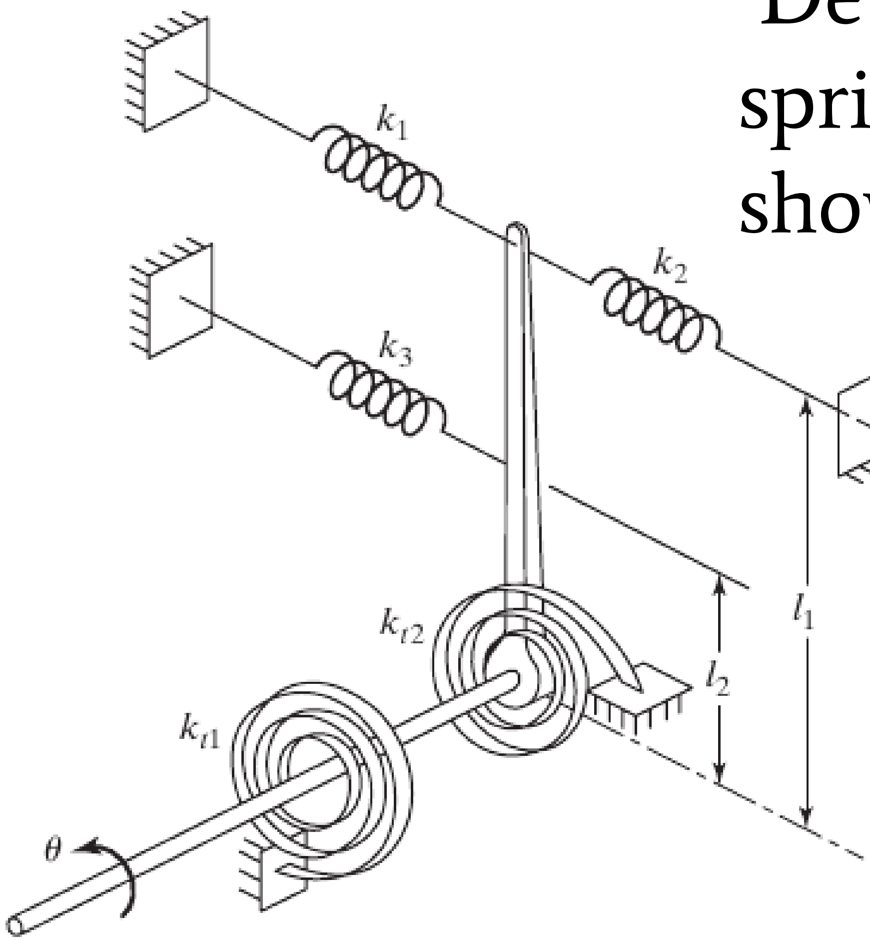
$$k_{\text{eq}} = k_1 \left(\frac{l_1}{l_3}\right)^2 + k_2 \left(\frac{l_2}{l_3}\right)^2$$

Problems



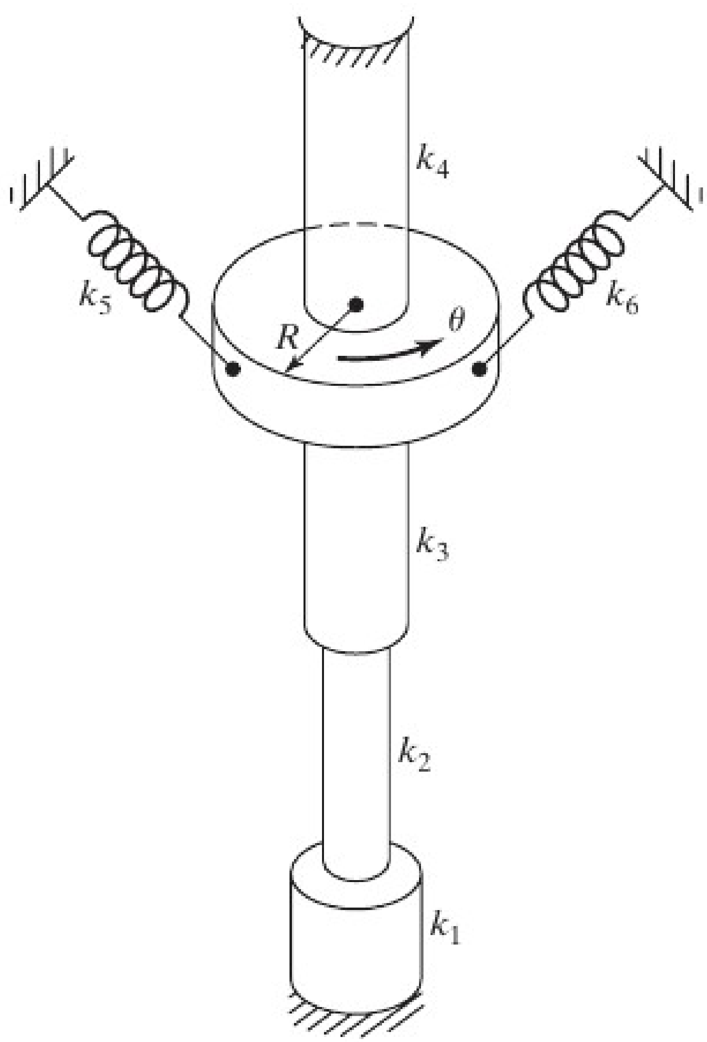
Determine the equivalent spring constant of the system shown in figure

Problems



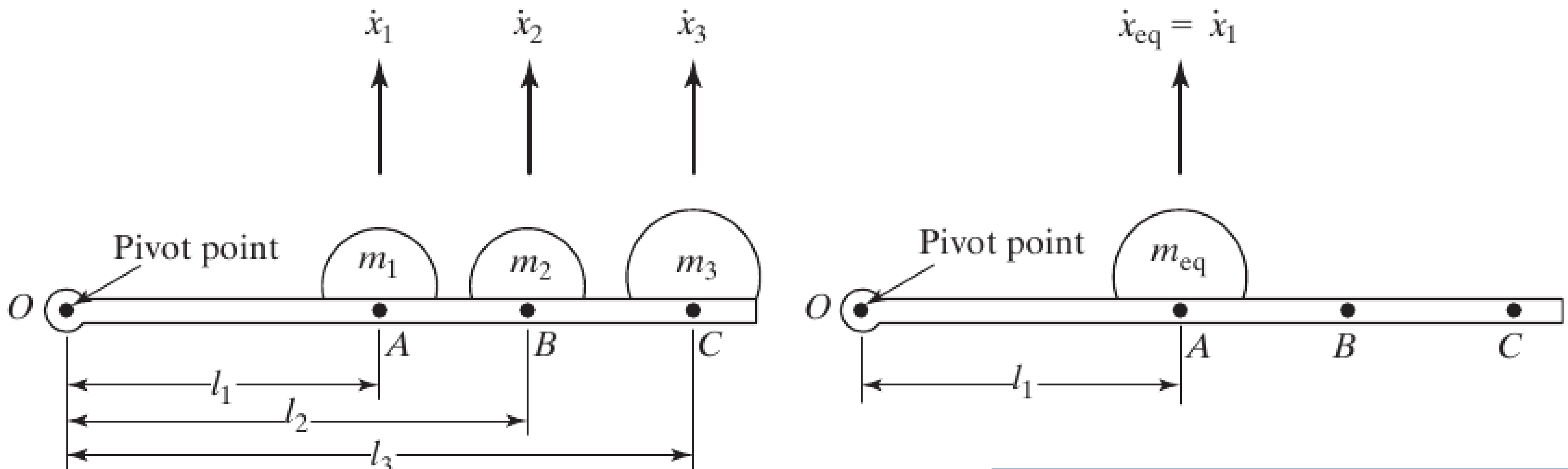
Determine the equivalent spring constant of the system shown in the figure

Problems



Determine the equivalent spring constant of the system shown in the figure

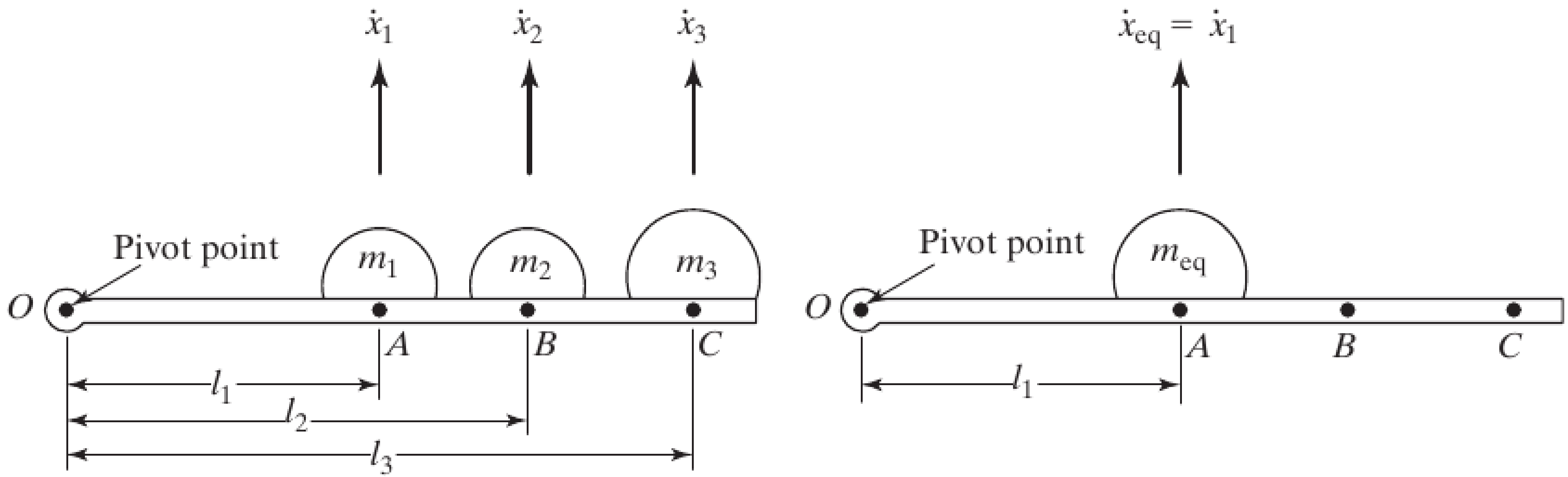
Mass or Inertia Elements



$$\sin(\theta) = \frac{x_1}{l_1} = \frac{x_2}{l_2} = \frac{x_3}{l_3} \longrightarrow \quad x_2 = \frac{l_2}{l_1}x_1 \qquad x_3 = \frac{l_3}{l_1}x_1$$

$$\dot{x}_2 = \frac{l_2}{l_1}\dot{x}_1 \qquad \dot{x}_3 = \frac{l_3}{l_1}\dot{x}_1$$

Mass or Inertia Elements

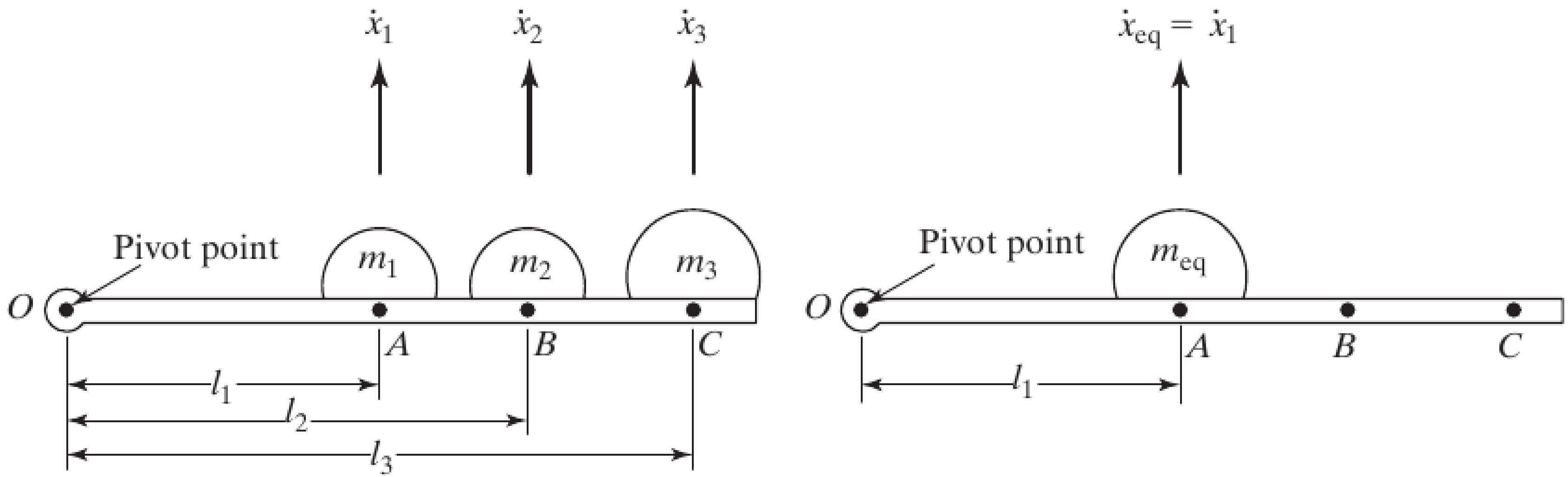


$$\dot{x}_{\mathrm{eq}} = \dot{x}_1$$

Kinetic Energy

$$\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 = \frac{1}{2}m_{\mathrm{eq}}\dot{x}_{\mathrm{eq}}^2$$

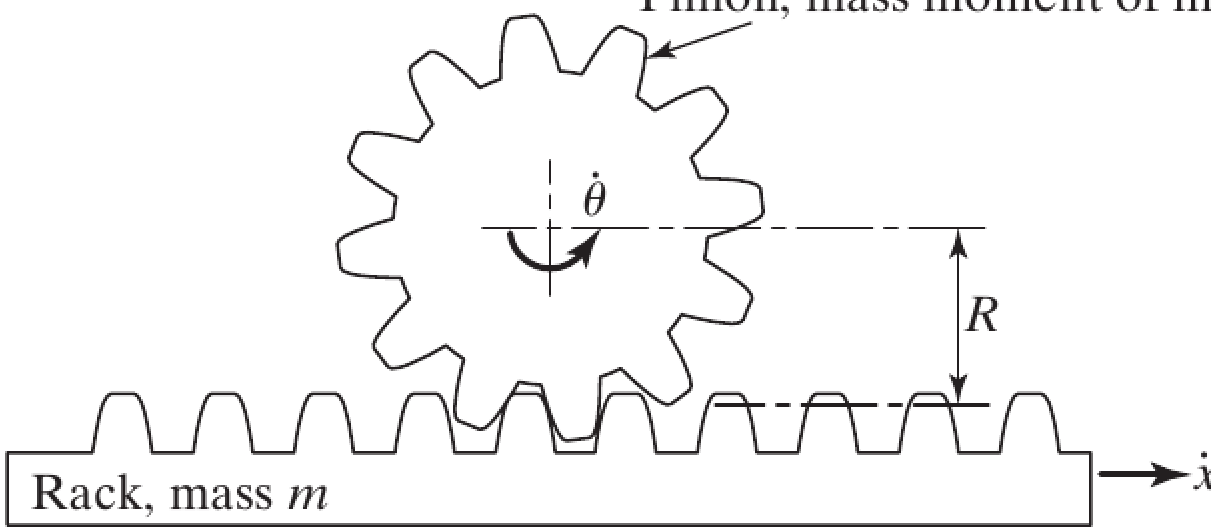
Mass or Inertia Elements



Equivalent Mass

$$m_{\text{eq}} = m_1 + m_2 \left(\frac{l_2}{l_1}\right)^2 + m_3 \left(\frac{l_3}{l_1}\right)^2$$

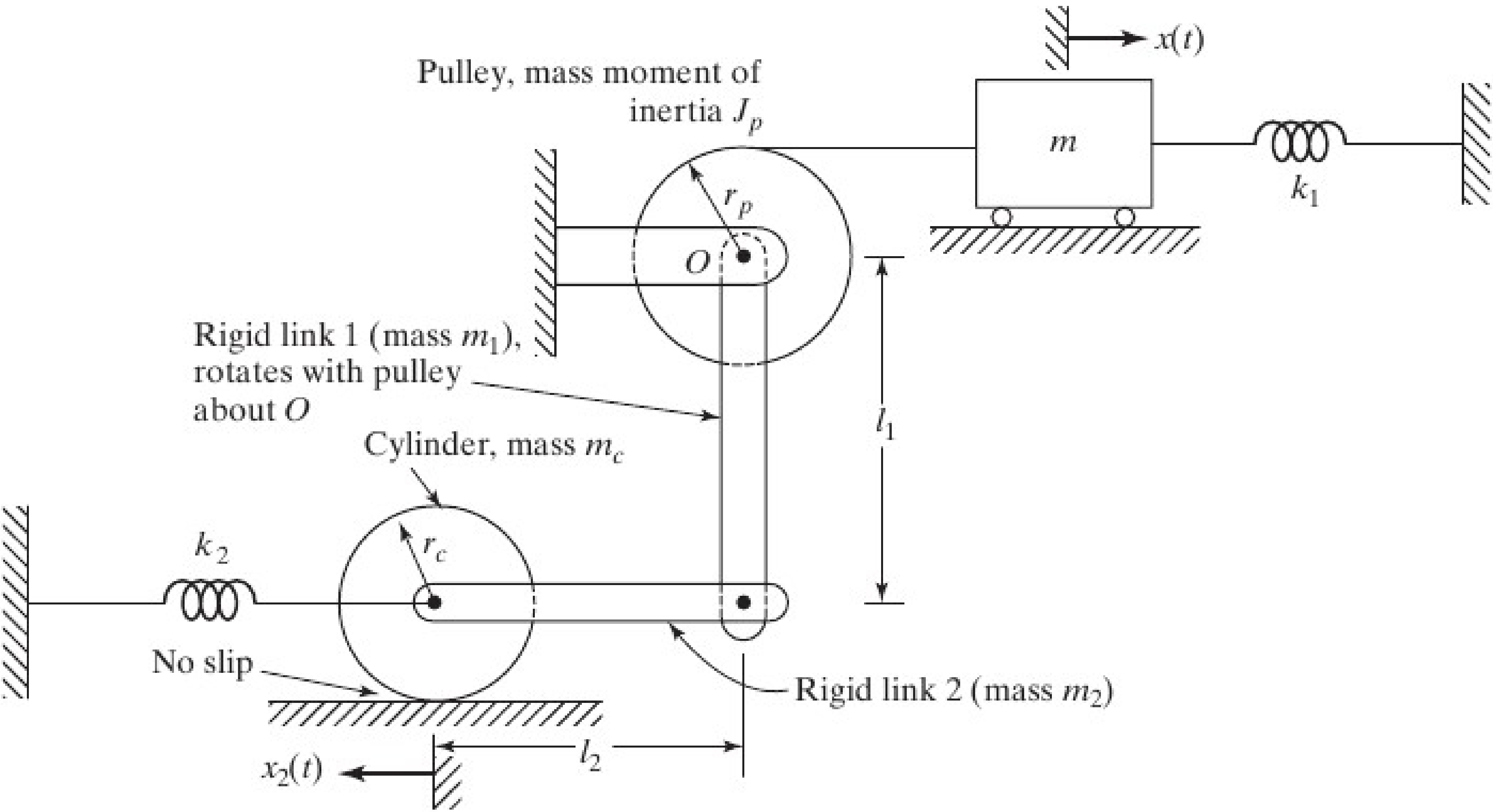
Other Examples



$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_0\dot{\theta}^2 \longrightarrow T_{eq} = \frac{1}{2}m_{eq}\dot{x}_{eq}^2$$

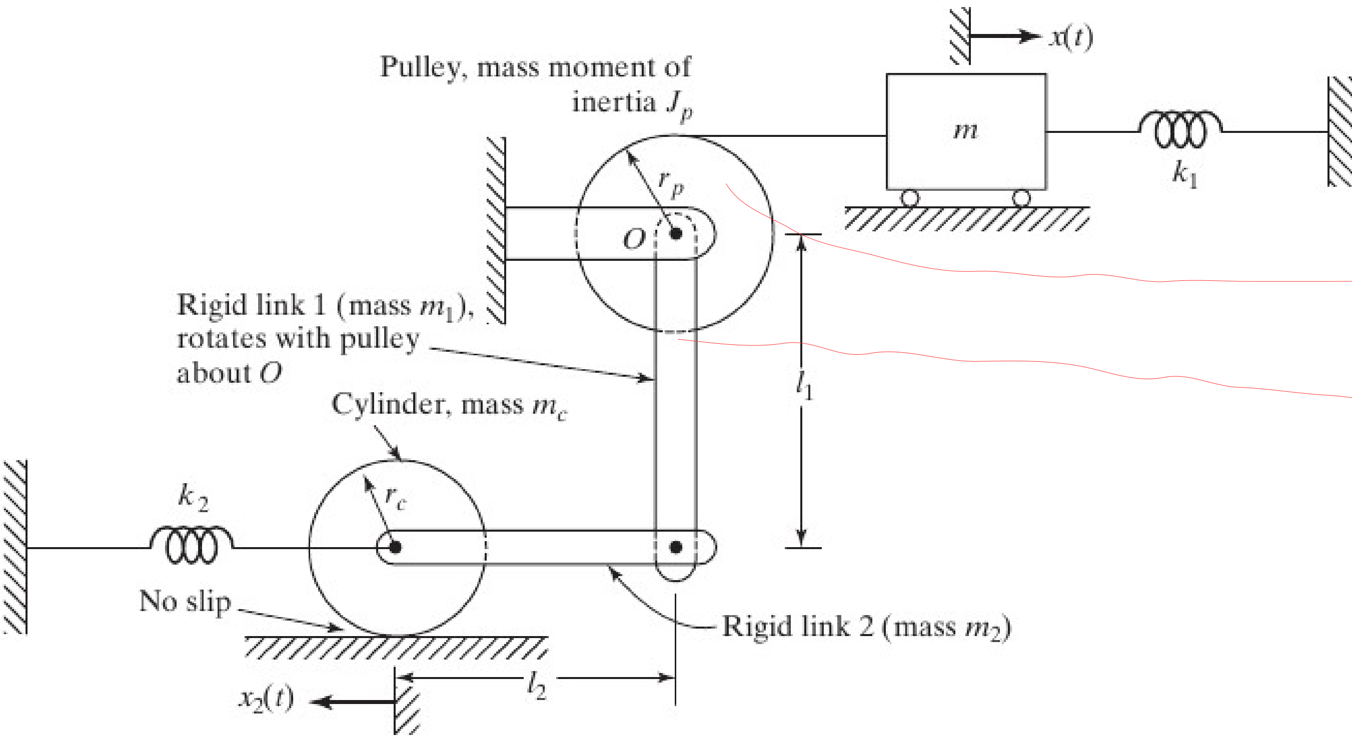
$$\frac{1}{2}m_{eq}\dot{x}^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_0\left(\frac{\dot{x}}{R}\right)^2 \longrightarrow m_{eq} = m + \frac{J_0}{R^2}$$

Example 1.6



x(t)
Pulley, mass moment of inertia J_p
m
k_1
r_p
O
Rigid link 1 (mass m_1), rotates with pulley about O
l_1
Cylinder, mass m_c
k_2
r_c
No slip
Rigid link 2 (mass m_2)
$x_2(t)$
l_2

Example 1.6



$$\theta_p = \frac{x}{r_p}$$

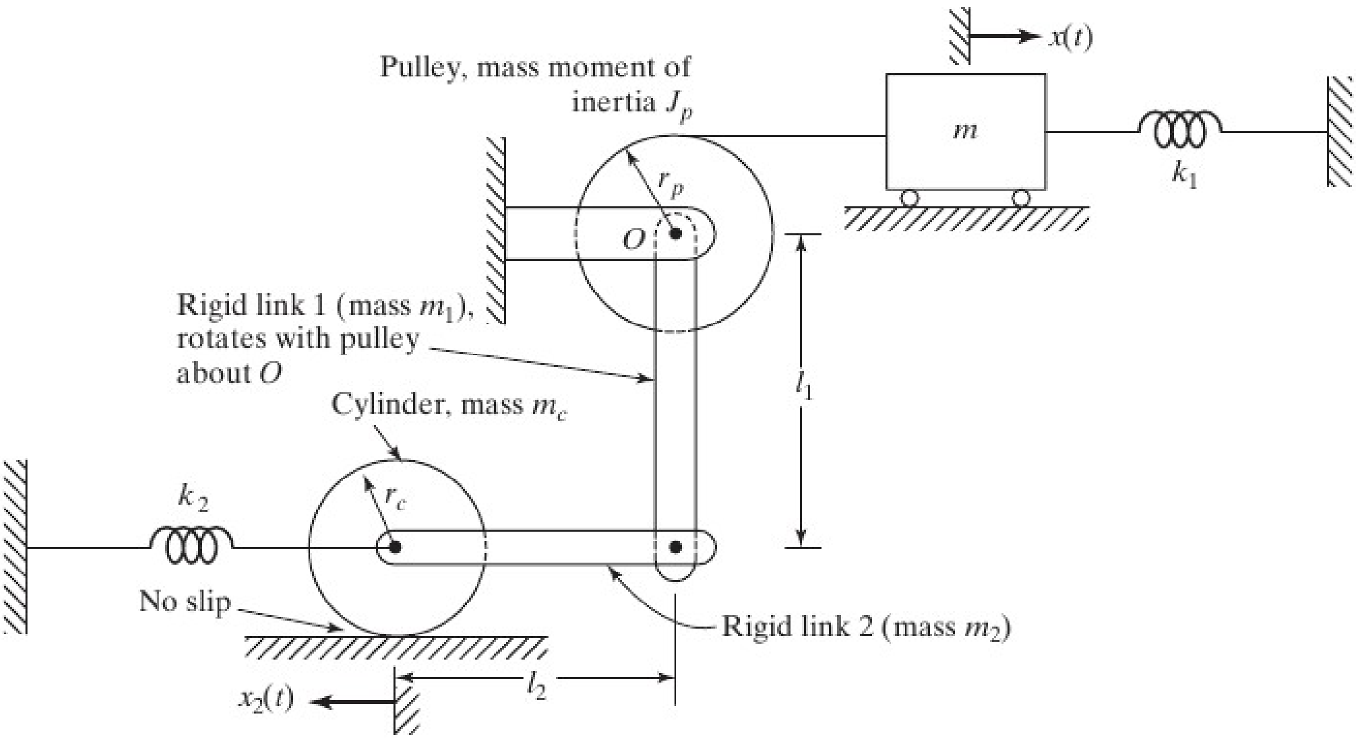
$$\theta_1 = \frac{x}{r_p}$$

$$x_2 = \theta_p l_1$$

$$\theta_c = \frac{x_2}{r_c}$$

$$\theta_c = \frac{x l_1}{r_p r_c}$$

Example 1.6



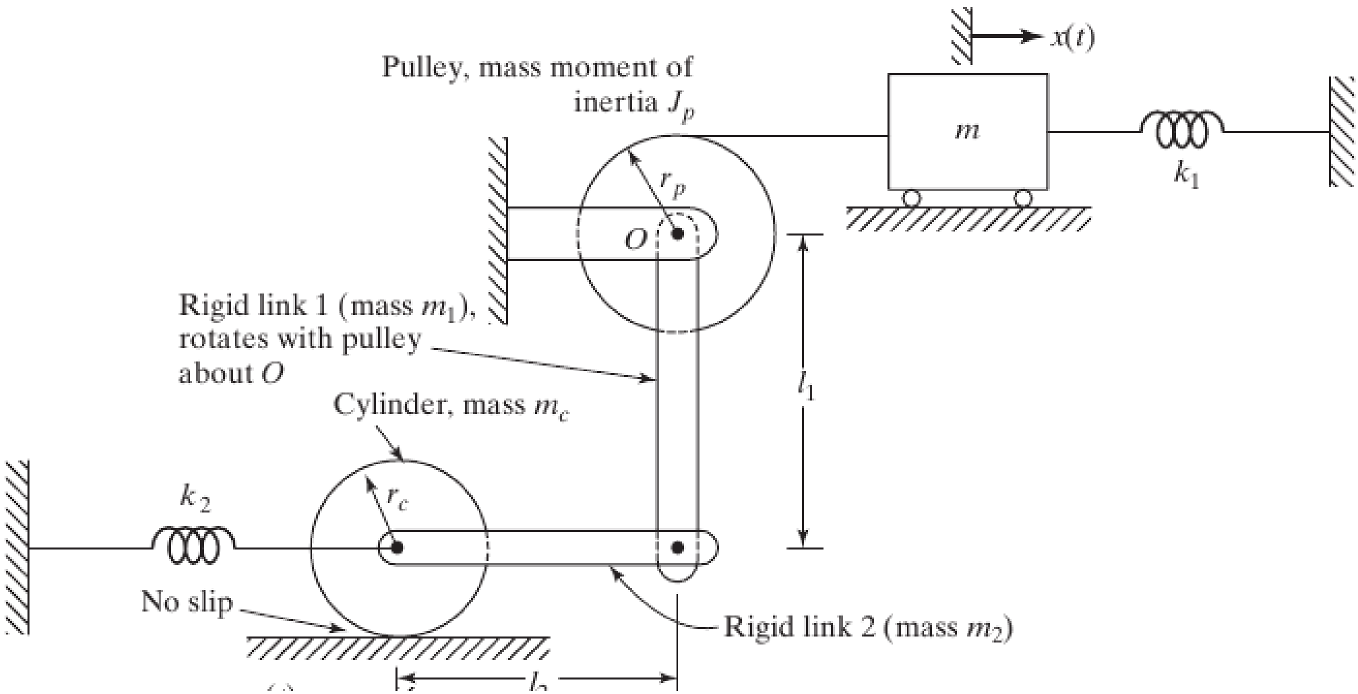
$$J_1 = \frac{m_1 l_1^2}{3}$$

$$J_c = \frac{m_c r_c^2}{2}$$

Kinetic Energy

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_p \dot{\theta}_p^2 + \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} J_c \dot{\theta}_c^2 + \frac{1}{2} m_c \dot{x}_2^2$$

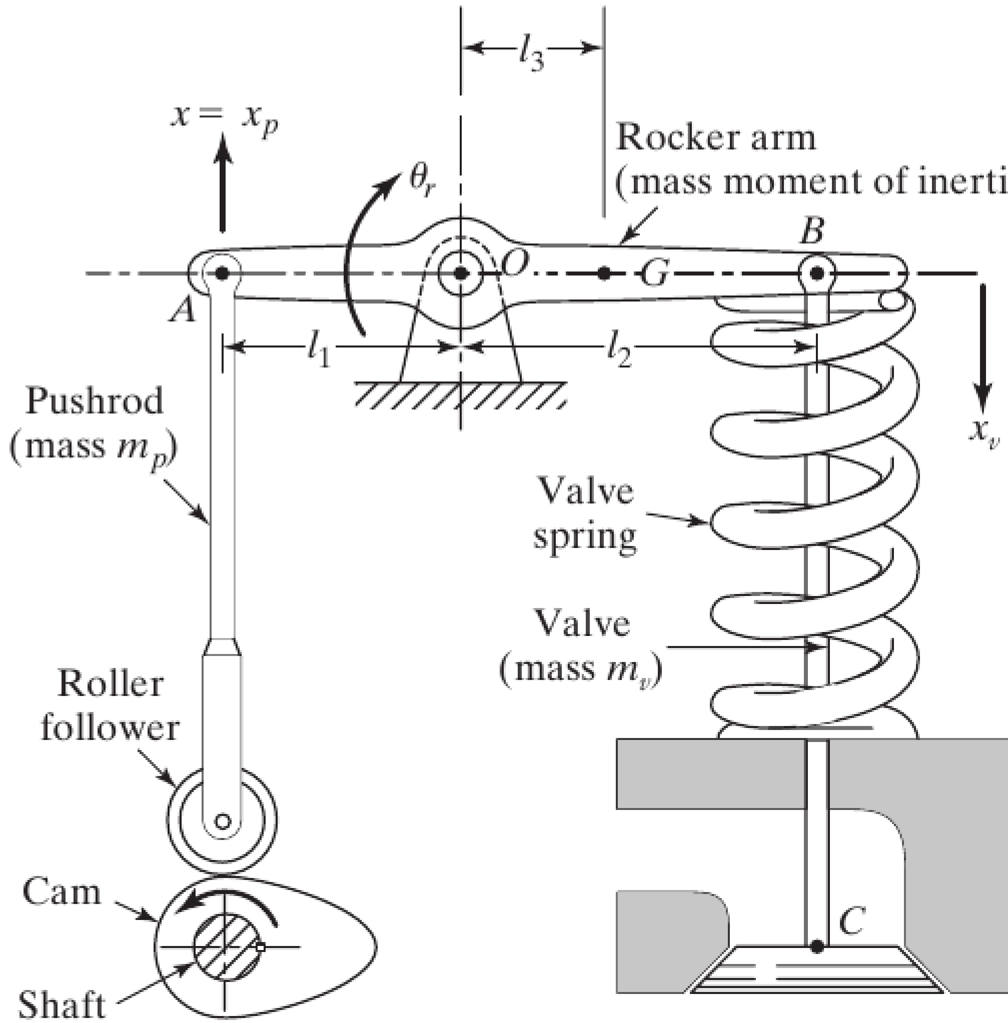
Example 1.6



$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J_p\left(\frac{\dot{x}}{r_p}\right)^2 + \frac{1}{2}\left(\frac{m_1 l_1^2}{3}\right)\left(\frac{\dot{x}}{r_p}\right)^2 + \frac{1}{2} m_2\left(\frac{\dot{x} l_1}{r_p}\right)^2$$

$$+ \frac{1}{2}\left(\frac{m_c r_c^2}{2}\right)\left(\frac{\dot{x} l_1}{r_p r_c}\right)^2 + \frac{1}{2} m_c\left(\frac{\dot{x} l_1}{r_p}\right)^2$$

Cam-Follower Mechanism (ex. 1.7)



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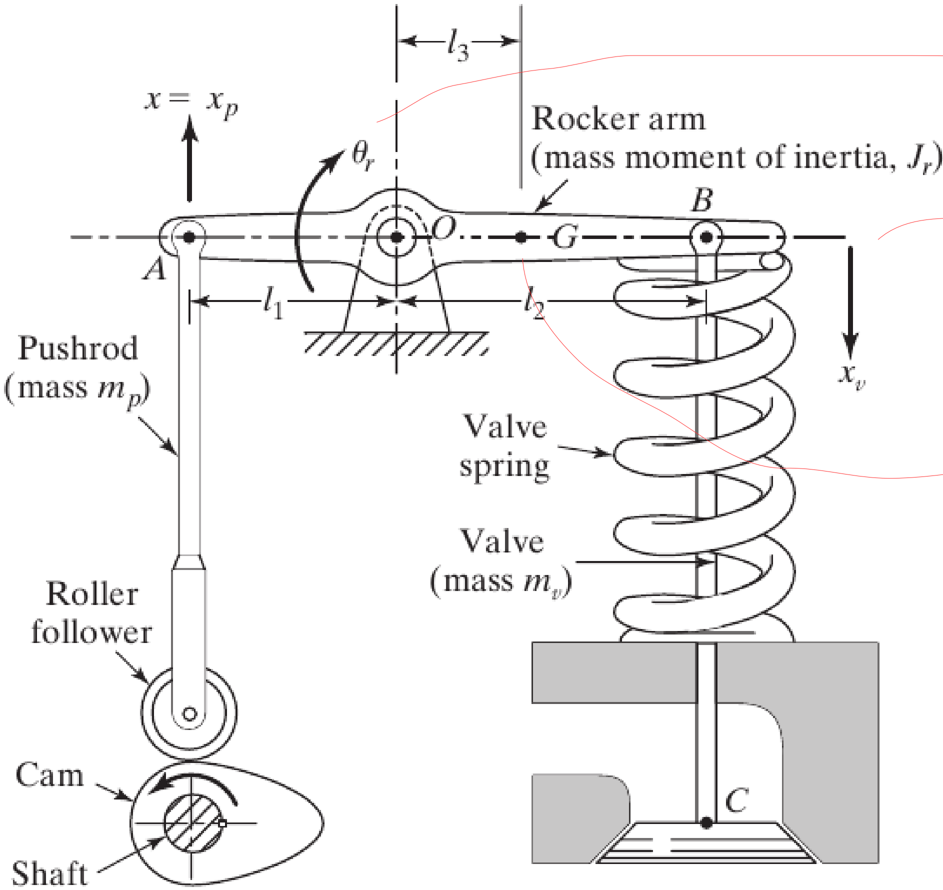


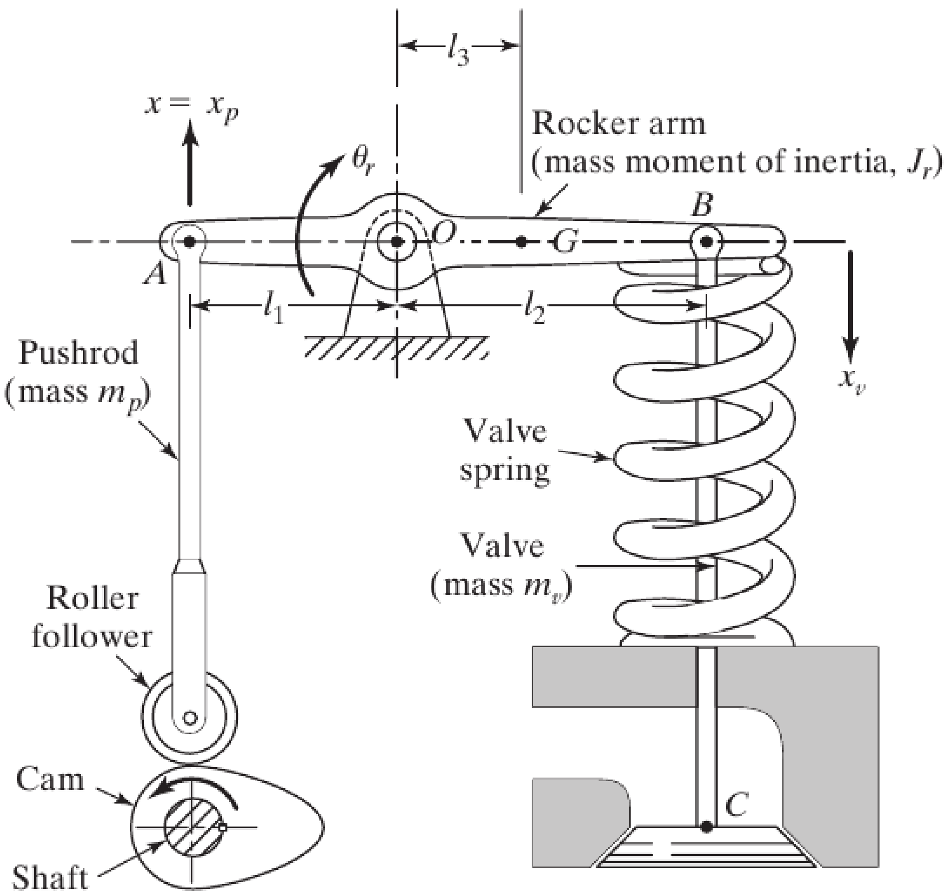
$$\theta_r = \frac{x}{l_1}$$

$$x_v = \theta_r l_2 \longrightarrow x_v = x\frac{l_2}{l_1}$$

$$x_r = \theta_r l_3 \longrightarrow x_r = x\frac{l_3}{l_1}$$

$$T = \frac{1}{2}m_p\dot{x}_p^2 + \frac{1}{2}m_v\dot{x}_v^2 + \frac{1}{2}J_r\dot{\theta}_r^2 + \frac{1}{2}m_r\dot{x}_r^2$$

Cam-Follower Mechanism (ex. 1.7)



$$T_{\text{eq}} = \frac{1}{2} m_{\text{eq}} \dot{x}_{\text{eq}}^2$$

$$\dot{x}_p = \dot{x}, \quad \dot{x}_v = \frac{\dot{x} l_2}{l_1},$$

$$\dot{x}_r = \frac{\dot{x} l_3}{l_1}, \quad \text{and} \quad \dot{\theta}_r = \frac{\dot{x}}{l_1}$$

$$T_{\text{eq}} = \frac{1}{2} m_{\text{eq}} \dot{x}_{\text{eq}}^2 = \frac{1}{2} m_{\text{eq}} \dot{x}_v^2$$

$$m_{\text{eq}} = m_p + \frac{J_r}{l_1^2} + m_v \frac{l_2^2}{l_1^2} + m_r \frac{l_3^2}{l_1^2}$$