Mechanical Vibrations

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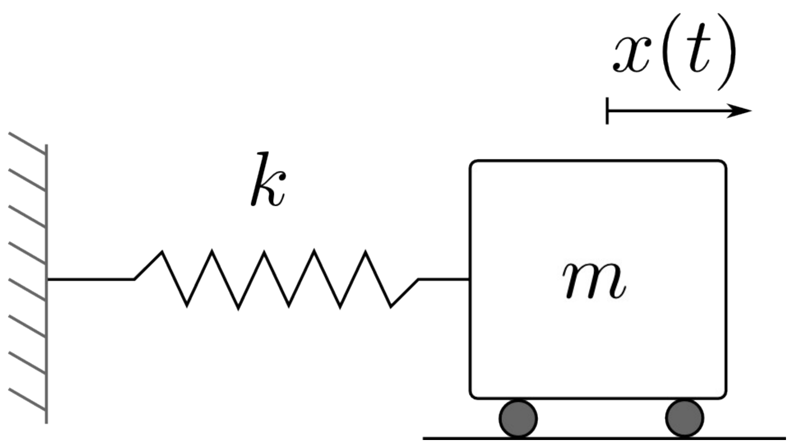
Equations of Motion



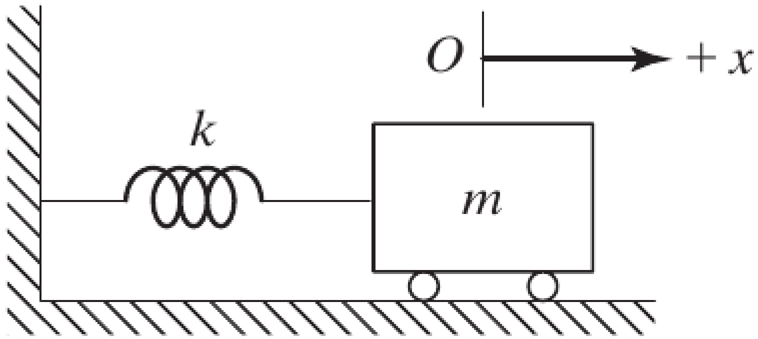
Newton's Method

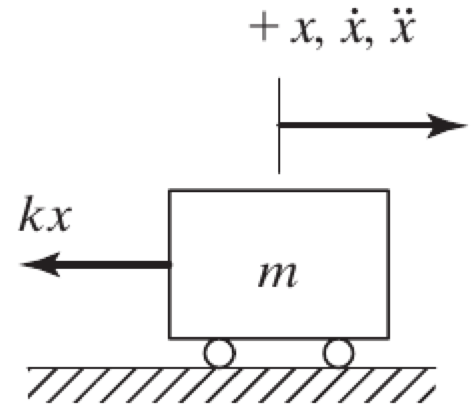
$$\sum F = m\ddot{x}(t)$$

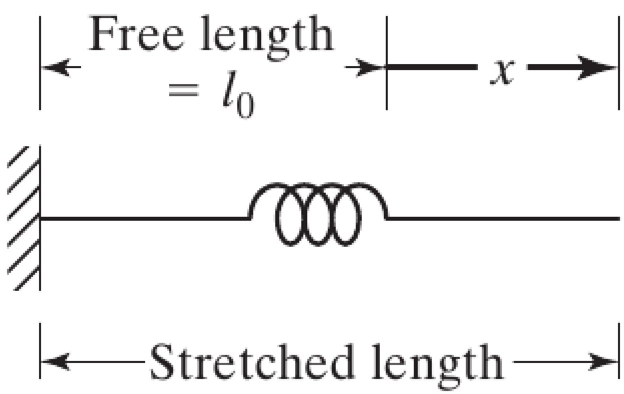
$$-kx(t) = m\ddot{x}(t)$$

$$m\ddot{x}(t) + kx(t) = 0$$

Equations of Motion







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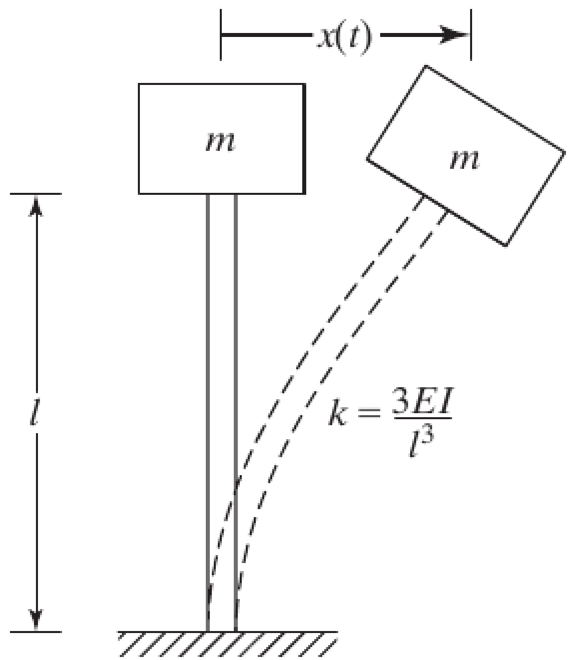
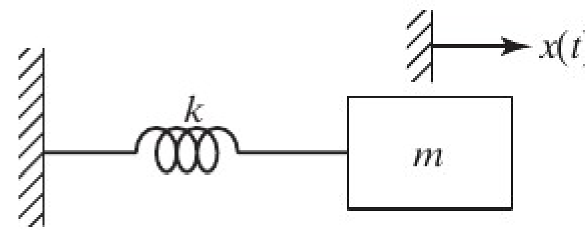


(a) Idealization of the tall structure



(b) Equivalent spring-mass system

Equations of Motion

Equation of Motion Using Other Methods

D'Alembert's Principle

$$\vec{F}(t) - m\ddot{\vec{x}} = 0$$

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Fictitious Inertia Force

$$-kx - m\ddot{x} = 0 \quad \text{or} \quad m\ddot{x} + kx = 0$$

Equations of Motion

Equation of Motion Using Other Methods

Principle of Virtual Displacements.

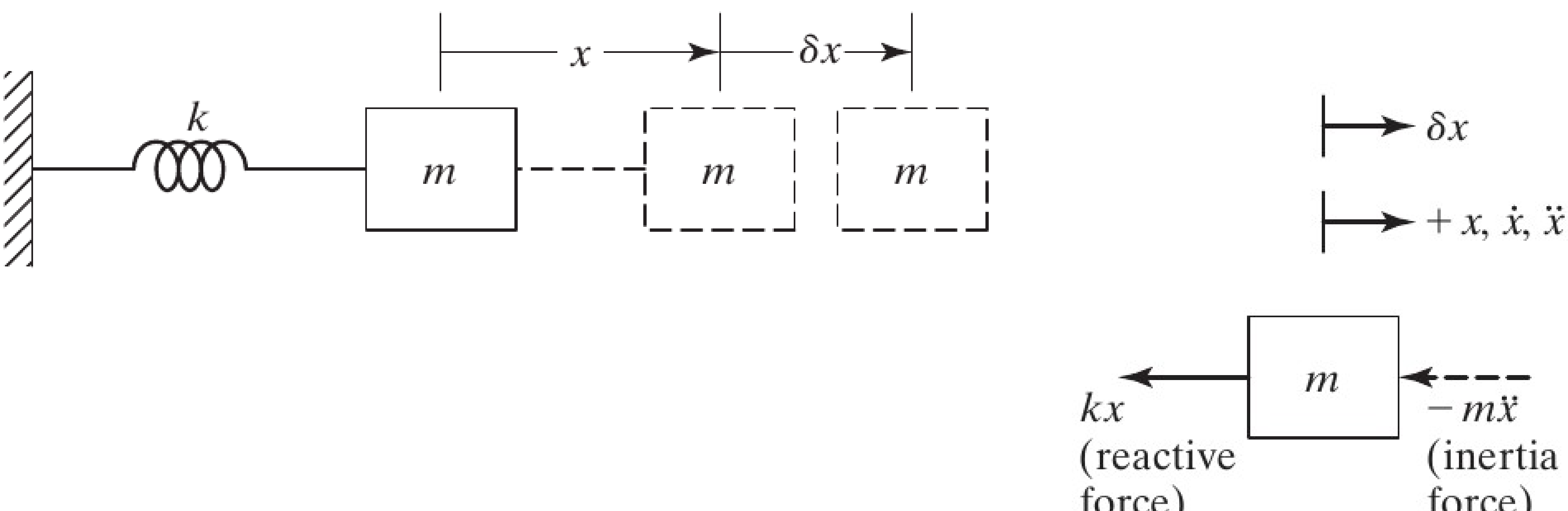
The principle of virtual displacements states that if a system that is in equilibrium under the action of a set of forces is subjected to a virtual displacement, then the total virtual work done by the forces will be zero. Here the virtual displacement is defined as an imaginary infinitesimal displacement given instantaneously.

Equations of Motion

Principle of Virtual Displacements.

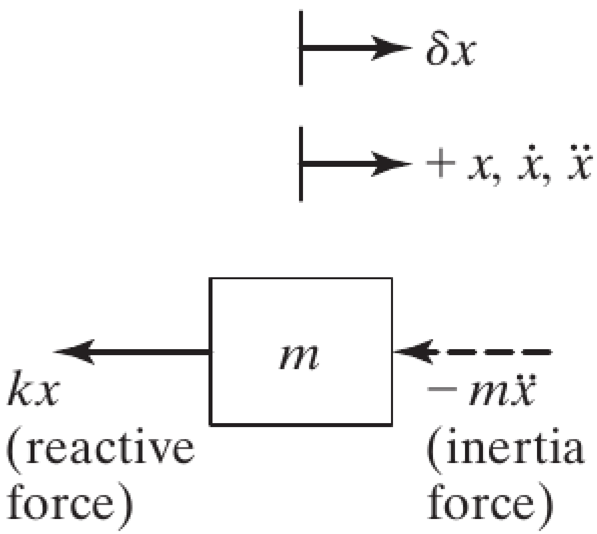
$$\text{Virtual work done by the spring force} = \delta W_s = -(kx)\,\delta x$$

$$\text{Virtual work done by the inertia force} = \delta W_i = -(m\ddot{x})\,\delta x$$



Equations of Motion

Principle of Virtual Displacements.



$$-m\ddot{x}\delta x - kx\,\delta x = 0$$

$$m\ddot{x} + kx = 0$$

Equations of Motion

Principle of Conservation of Energy.

$$T + U = \text{Constant}$$

Kinetic Energy

Potential Energy

$$\frac{d}{dt}\left(T + U\right) = 0$$

Equations of Motion

$$\frac{d}{dt}\left(T+U\right)=0 \longrightarrow \begin{aligned} T &= \frac{1}{2}m\dot{x}^2 \\ U &= \frac{1}{2}kx^2 \end{aligned}$$

$$\dot{x}\left[kx+m\ddot{x}\right]=0$$

Obtaining the Natural Frequency

The energy method can be used to obtain the natural frequency by considering

$$T_{\max} = \frac{1}{2} m \dot{x}_{\max}^2 \qquad U_{\max} = \frac{1}{2} k x_{\max}^2$$

$$T_{\max} = U_{\max}$$

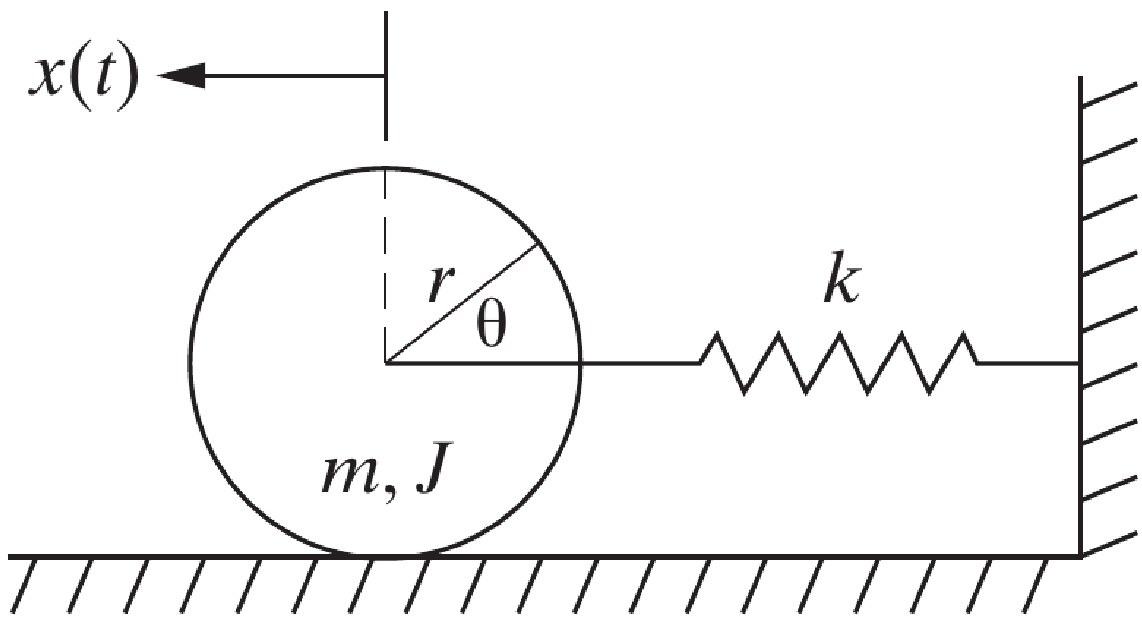
$$x_{\max} = A$$
$$\dot{x}_{\max} = \omega_n A$$

$$\frac{1}{2} m (\omega_n A)^2 = \frac{1}{2} k A^2$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

Example 1

Calculate the natural frequency using the energy method. Assume no dissipation of energy.



$$T_{\text{rot}} = \frac{1}{2}J\dot{\theta}^2 \qquad T_T = \frac{1}{2}m\dot{x}^2 \qquad U = \frac{1}{2}kx^2 \qquad x = r\theta$$

Example 1 - Solution

$$T_{\max} = \frac{1}{2} m\dot{x}_{\max}^2 + \frac{1}{2}\frac{J}{r^2}\dot{x}_{\max}^2 = \frac{1}{2}(m + J/r^2)\omega_n^2 A^2$$

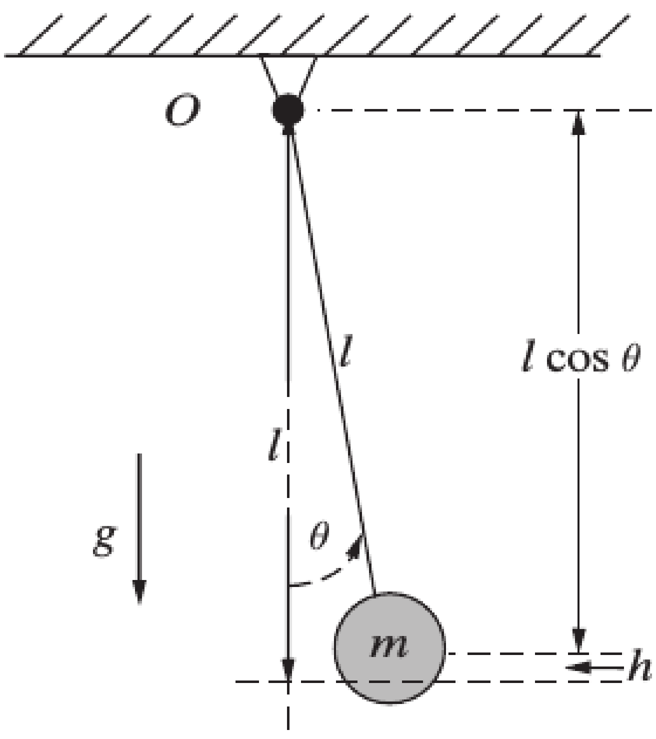
$$U_{\max} = \tfrac{1}{2} k x_{\max}^2 = \tfrac{1}{2} k A^2$$

$$\frac{1}{2}\left(m + \frac{J}{r^2}\right)\omega_n^2 = \frac{1}{2} k$$

$$\omega_n = \sqrt{\frac{k}{m + J/r^2}}$$

Example 2

Calculate the equations of motion using the energy method. Assume no dissipation of energy.



$$J = ml^2$$

$$T = \frac{1}{2} J\dot{\theta}^2 = \frac{1}{2} ml^2\dot{\theta}^2$$

$$U = mgl(1 - \cos\theta)$$

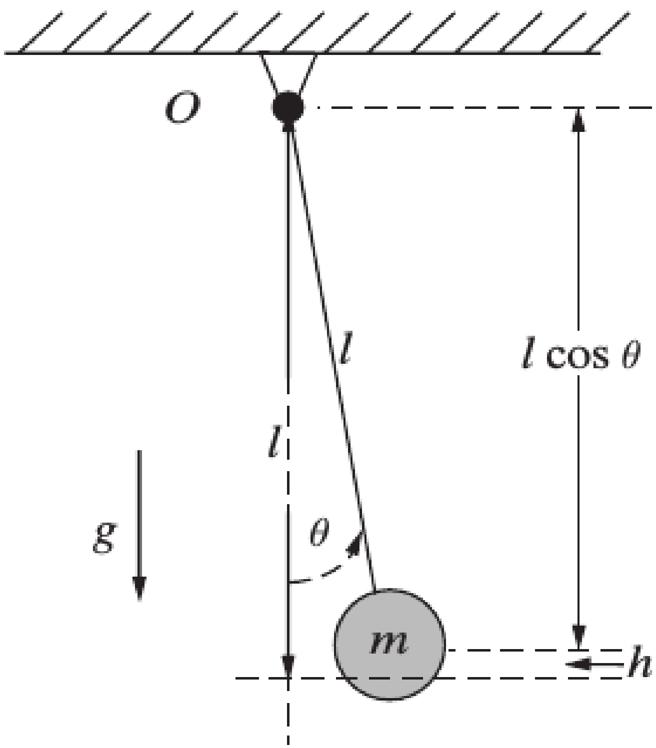
Example 2 - Solution

$$\frac{d}{dt}\left[\tfrac{1}{2}ml^2\dot{\theta}^2 + mgl(1 - \cos\theta)\right] = 0$$

$$ml^2\dot{\theta}\ddot{\theta} + mgl(\sin\theta)\dot{\theta} = 0$$

$$\dot{\theta}(ml^2\ddot{\theta} + mgl\sin\theta) = 0$$

Example 2 - Solution

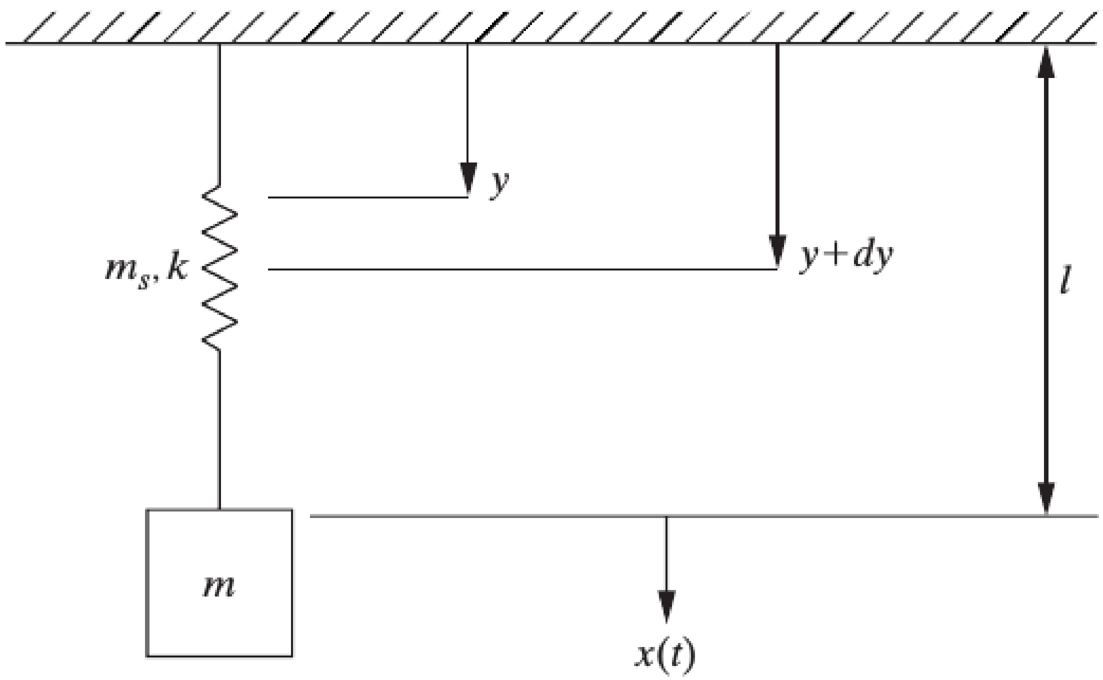


$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Example 3

Model the mass of the spring in the system shown in the figure. determine the effect of including the mass of the spring on the value of the natural frequency.



Example 3 - Solution

One approach to considering the mass of the spring in analysing the system vibration response is to calculate the kinetic energy of the spring.

Consider the kinetic energy of the element

$$dy$$

of the spring. If m_s is the total mass of the spring, then

$$\frac{m_s}{l}dy$$

is the mass of the element

Example 3 - Solution

A expression for the velocity of the element can be written assuming that the velocity varies linearly with the length of the spring

$$v_{dy} = \frac{y}{l}\dot{x}$$

The total kinetic energy of the spring is the kinetic energy of the element dy integrated over the length of the spring:

$$T_{\text{spring}} = \frac{1}{2}\int_0^l \frac{m_s}{l}\left[\frac{y}{l}\dot{x}\right]^2 dy \qquad = \frac{1}{2}\left(\frac{m_s}{3}\right)\dot{x}^2$$

Example 3 - Solution

Following the energy method, the maximum kinetic energy of the system is thus

$$T_{\max} = \frac{1}{2}\left(m + \frac{m_s}{3}\right)\omega_n^2 A^2$$

The natural frequency can be calculated by

$$\omega_n = \sqrt{\frac{k}{m + m_s/3}}$$