Mechanical Vibrations

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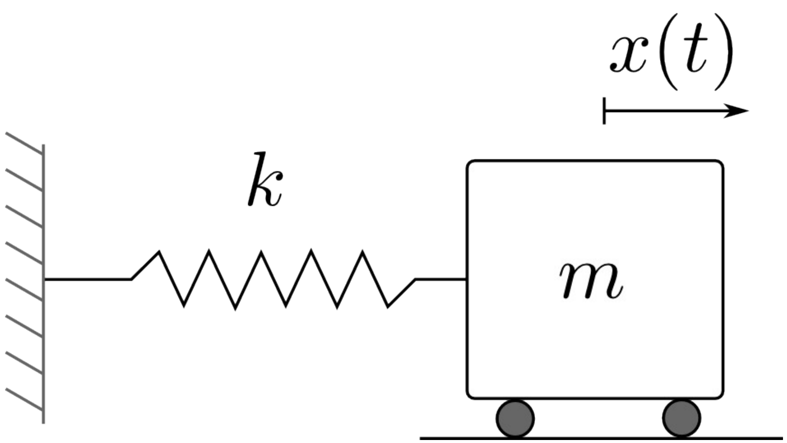
Equations of Motion



Newton's Method

$$\sum F = m\ddot{x}(t)$$

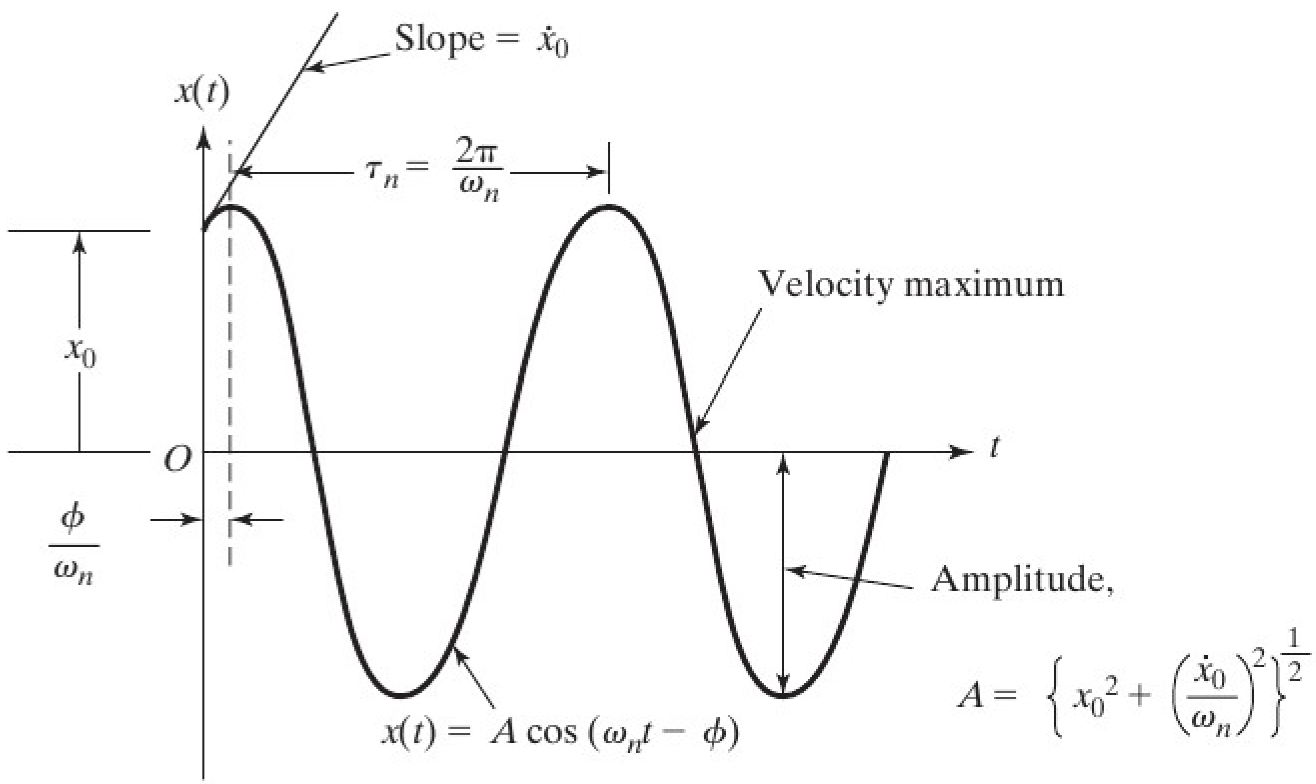
$$-kx(t) = m\ddot{x}(t)$$

$$m\ddot{x}(t) + kx(t) = 0$$

Undamped Vibration (sdof)

$$x(t) = x_0 \cos(\omega_n t) + \frac{v_0}{\omega_n} \sin(\omega_n t)$$

$$x(t) = A \sin(\omega_n t + \phi)$$



Displacement – Velocity Plane

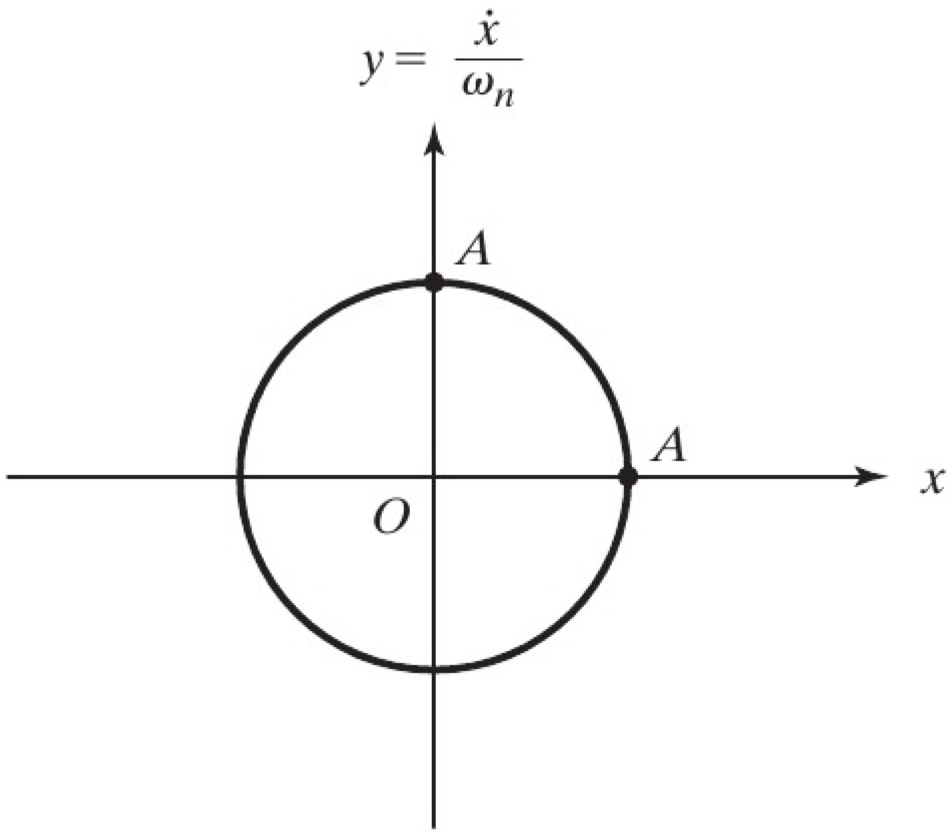
$$x(t) = A\cos(\omega_n t - \phi) \longrightarrow \cos(\omega_n t - \phi) = \frac{x}{A}$$

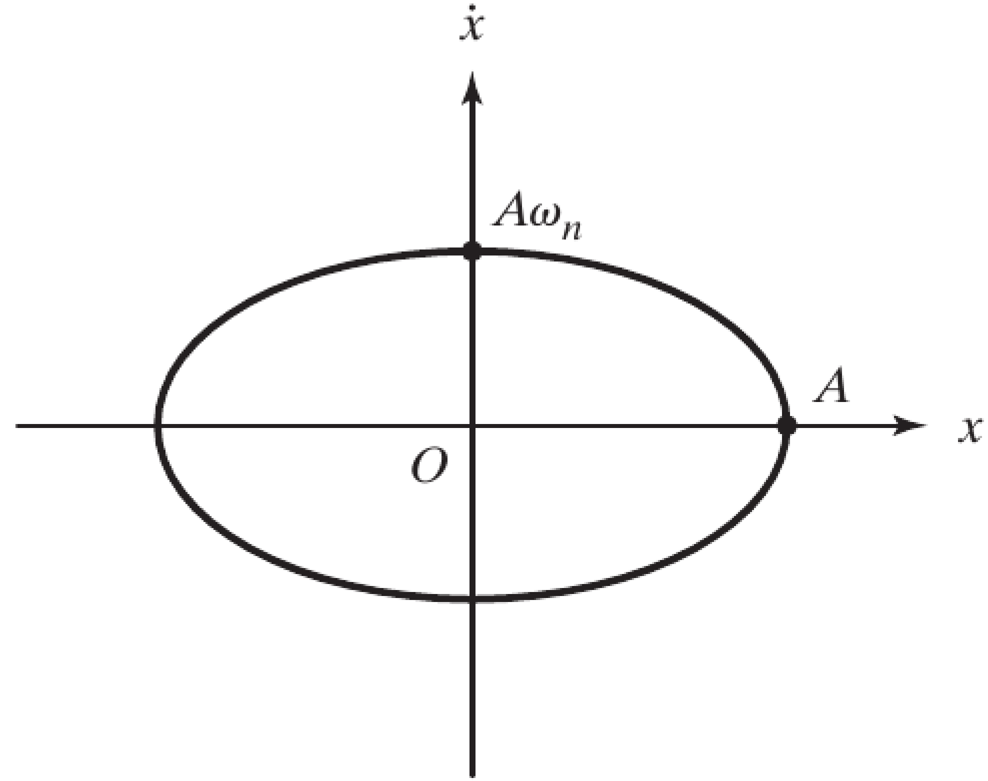
$$\dot{x}(t) = -A\omega_n \sin(\omega_n t - \phi) \longrightarrow \sin(\omega_n t - \phi) = -\frac{\dot{x}}{A\omega_n} = -\frac{y}{A}$$

$$\cos^2(\omega_n t - \phi) + \sin^2(\omega_n t - \phi) = 1$$

$$\frac{x^2}{A^2} + \frac{y^2}{A^2} = 1$$

Displacement – Velocity Plane





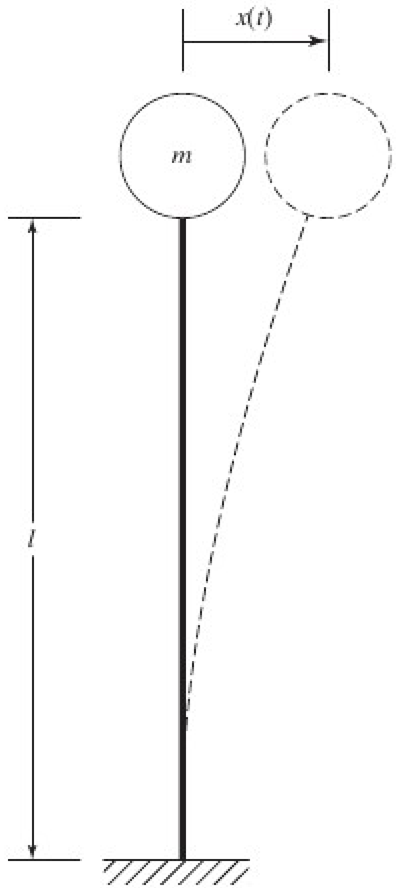
Example 2.1

Harmonic Response of a water tank

The column of the water tank shown in the figure is 300 ft high and is made of reinforced concrete with a tubular cross section of inner diameter 8 ft and outer diameter 10 ft. The tank weighs 6×10^5 lb when filled with water. By neglecting the mass of the column and assuming the Young s modulus of reinforced concrete as 4×10^6 psi, determine the following:

Example 2.1

Harmonic Response of a water tank



Question a) Find the natural frequency and the natural time period of transverse vibration of the water tank.

Example 2.1

$$k = \frac{P}{\delta} = \frac{3EI}{l^3}$$

$$l = 3600 \text{ in.}, \; E = 4 \times 10^6 \text{ psi},$$

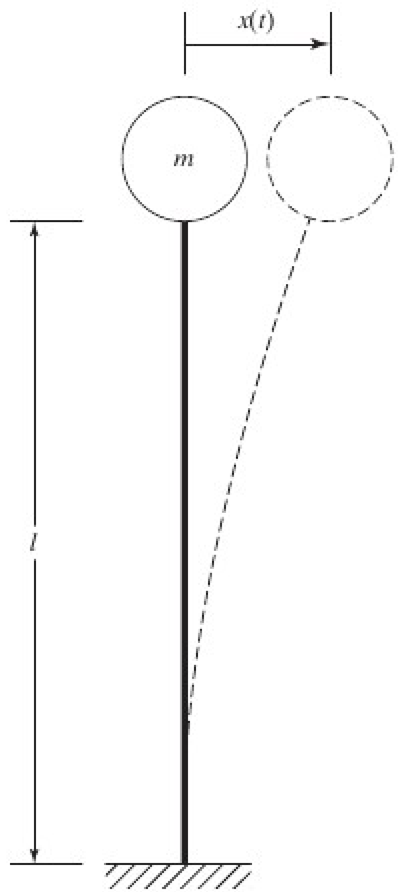
$$I = \frac{\pi}{64}(d_0^4 - d_i^4) = \frac{\pi}{64}(120^4 - 96^4)$$

$$= 600.9554 \times 10^4 \text{ in.}^4$$

$$k = \frac{3(4 \times 10^6)(600.9554 \times 10^4)}{3600^3}$$

$$= 1545.6672 \text{ lb/in.}$$

Example 2.1



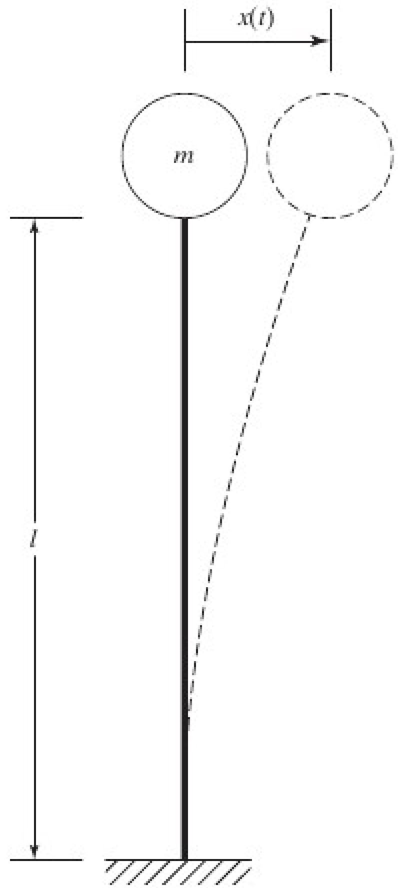
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1545.6672 \times 386.4}{6 \times 10^5}}$$

$$= 0.9977 \text{ rad/sec}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{0.9977} = 6.2977 \text{ sec}$$

Example 2.1

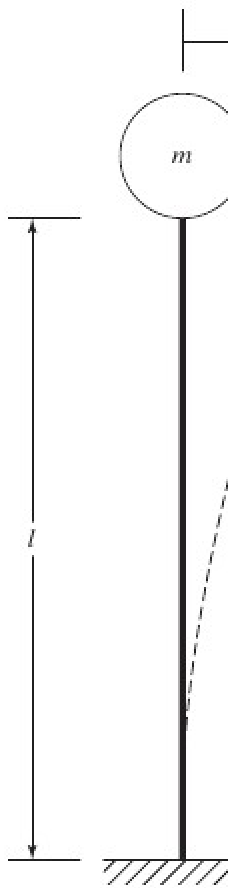
Harmonic Response of a water tank



Question b) Find the the vibration response of the water tank due to an initial transverse displacement of 10 in.

Example 2.1

Harmonic Response of a water tank



$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

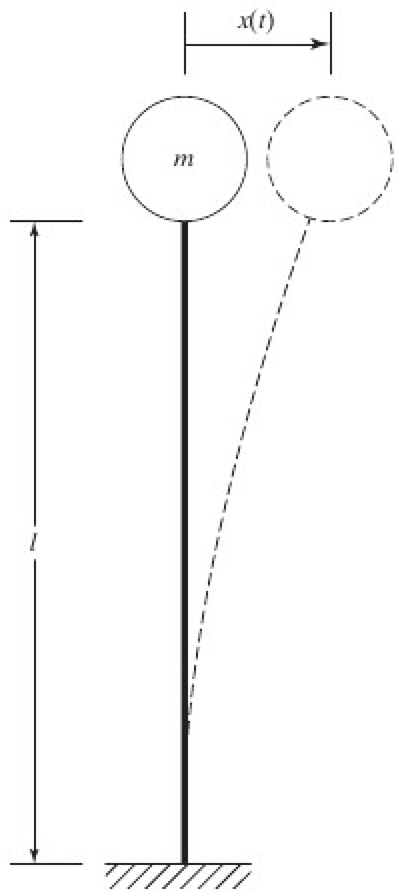
$$A_0 = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2} = x_0 = 10 \text{ in.}$$

$$\phi_0 = \tan^{-1}\left(\frac{x_0 \omega_n}{0} \right) = \frac{\pi}{2}$$

$$x(t) = 10 \sin\left(0.9977t + \frac{\pi}{2} \right) = 10 \cos 0.9977t \text{ in.}$$

Example 2.1

Harmonic Response of a water tank



Question c) Find the maximum values of the velocity and acceleration experienced by the water tank.

Example 2.1

Harmonic Response of a water tank

$$x(t) = 10 \sin\left(0.9977t + \frac{\pi}{2}\right) = 10 \cos 0.9977t \text{ in.}$$

$$\dot{x}(t) = 10(0.9977) \cos\left(0.9977t + \frac{\pi}{2}\right)$$

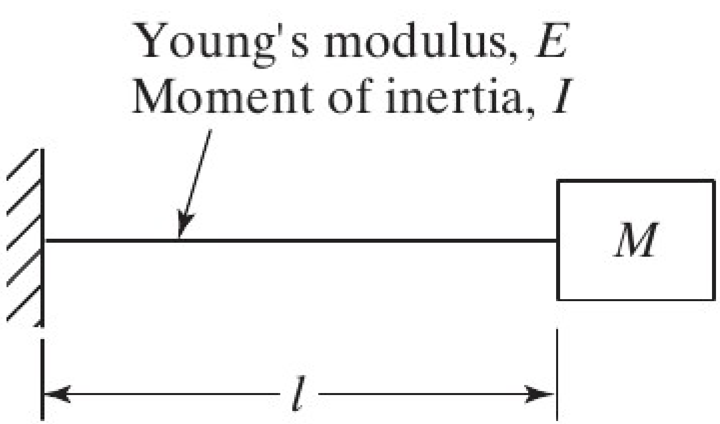
$$\dot{x}_{\max} = A_0\omega_n = 10(0.9977) = 9.977 \text{ in./sec}$$

$$\ddot{x}(t) = -10(0.9977)^2 \sin\left(0.9977t + \frac{\pi}{2}\right)$$

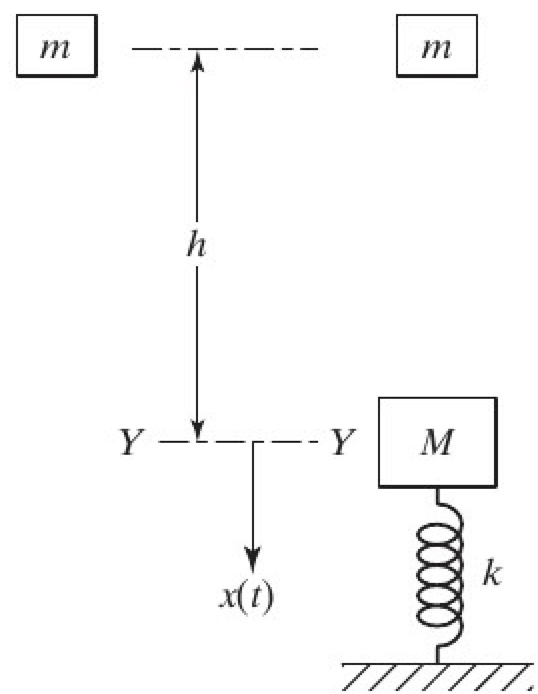
$$\ddot{x}_{\max} = A_0(\omega_n)^2 = 10(0.9977)^2 = 9.9540 \text{ in./sec}^2$$

Response due to Impact – Ex. 2.2

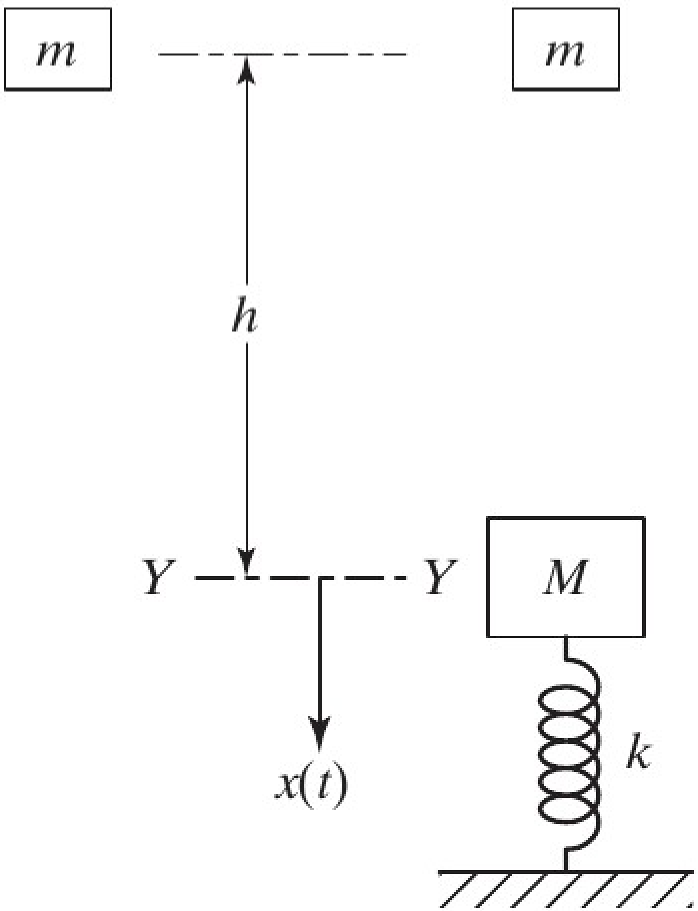
A cantilever beam carries a mass M at the free end as shown. A mass m falls from a height h onto the mass M and adheres to it without rebounding. Determine the resulting transverse vibration of the beam.



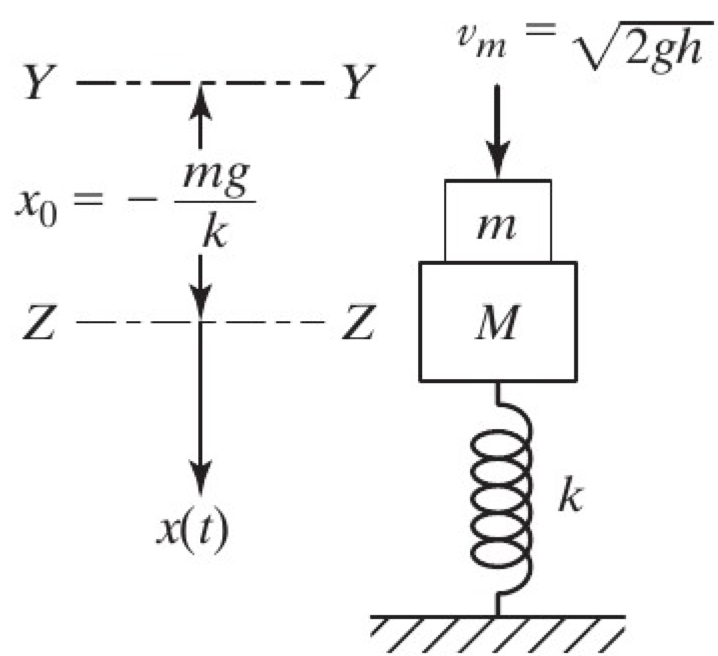
$$k = \frac{3EI}{l^3}$$



Response due to Impact – Solution



YY = static equilibrium position of M
ZZ = static equilibrium position of $M + m$



Response due to Impact – Solution

Conservation of Momentum

$$mv_m = (M + m)\dot{x}_0$$

$$\dot{x}_0 = \left(\frac{m}{M + m}\right)v_m = \left(\frac{m}{M + m}\right)\sqrt{2gh}$$

Response due to Impact – Solution

Since free vibration of the beam with the new mass (M + m) occurs about its own static equilibrium position, the initial conditions of the problem can be stated as

$$x_0 = -\frac{mg}{k}, \qquad \dot{x}_0 = \left(\frac{m}{M + m}\right)\sqrt{2gh} \qquad k = \frac{3EI}{l^3}$$

$$x(t) = A\cos(\omega_n t - \phi) \qquad A = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right]^{1/2} \qquad \phi = \tan^{-1}\left(\frac{\dot{x}_0}{x_0\omega_n}\right)$$

$$\omega_n = \sqrt{\frac{k}{M + m}} = \sqrt{\frac{3EI}{l^3(M + m)}}$$

Problem 2.6

The maximum velocity attained by the mass of a simple harmonic oscillator is 10 cm/s, and the period of oscillation is 2 s. If the mass is released with an initial displacement of 2 cm, find

(a) the amplitude
(b) the initial velocity
(c) the maximum acceleration
(d) the phase angle

Problem 2.6

$$x = A\cos\left(\omega_n t - \phi_0\right)$$

$$\dot{x} = -\omega_n A\sin\left(\omega_n t - \phi_0\right)$$

$$\dot{x} = -\omega_n^2 A\cos\left(\omega_n t - \phi_0\right)$$

$$\omega_n A = 0.1$$

$$\tau_n = \frac{2\pi}{\omega_n} = 2sec. \longrightarrow \omega_n = 3.1416\, rad/sec$$

$$A = 0.03183\, m$$ (a)

Problem 2.6

(d) the phase angle

$$x_0 = x(t = 0) = A \cos(-\phi_0) = 0.02\,m$$

$$\cos(-\phi_0) = \frac{0.02}{A} = 0.6283$$

$$\phi_0 = 51.0724^o$$

Problem 2.6

(b) the initial velocity

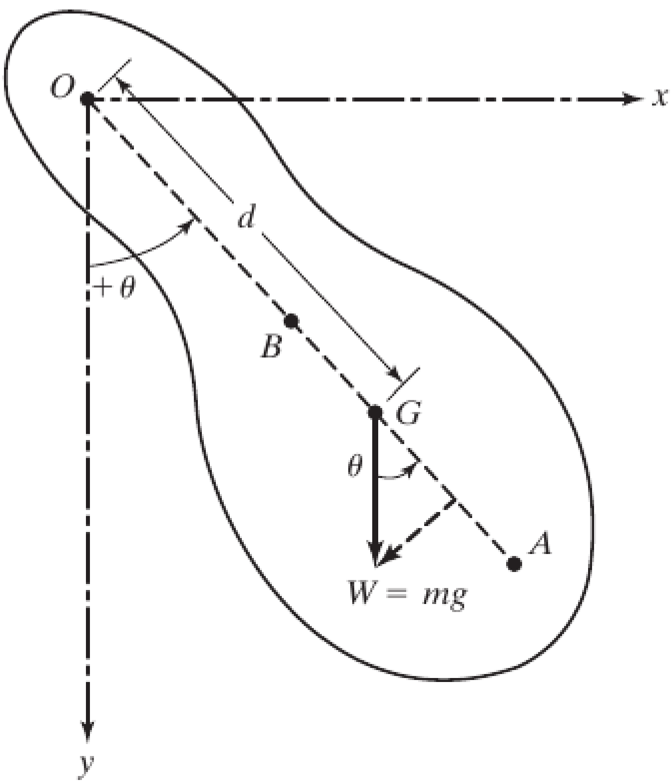
$$\dot{x}_0 = \dot{x}(t=0) = -\omega_n A \sin(-\phi_0) = -0.1 \sin(-51.0724^o)$$

$$\dot{x}_0 = 0.07779\, m/sec$$

(c) the maximum acceleration

$$\ddot{x}_{max} = \omega_n^2 A = 0.314151 m/sec^2$$

Compound Pendulum



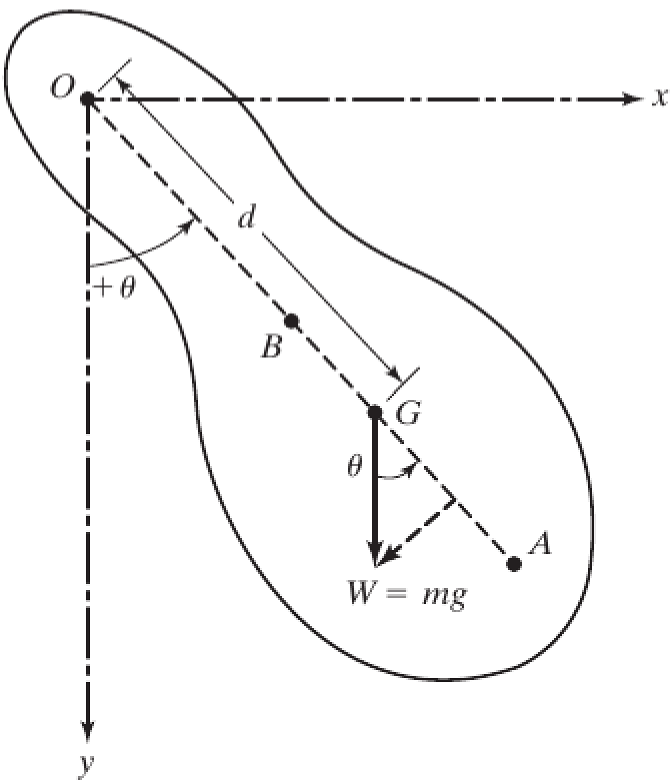
$$J_0\ddot{\theta} + Wd\sin\theta = 0$$

Small angles: $\sin\theta \approx \theta$

$$J_0\ddot{\theta} + Wd\theta = 0$$

$$\omega_n = \left(\frac{Wd}{J_0}\right)^{1/2} = \left(\frac{mgd}{J_0}\right)^{1/2}$$

Compound Pendulum



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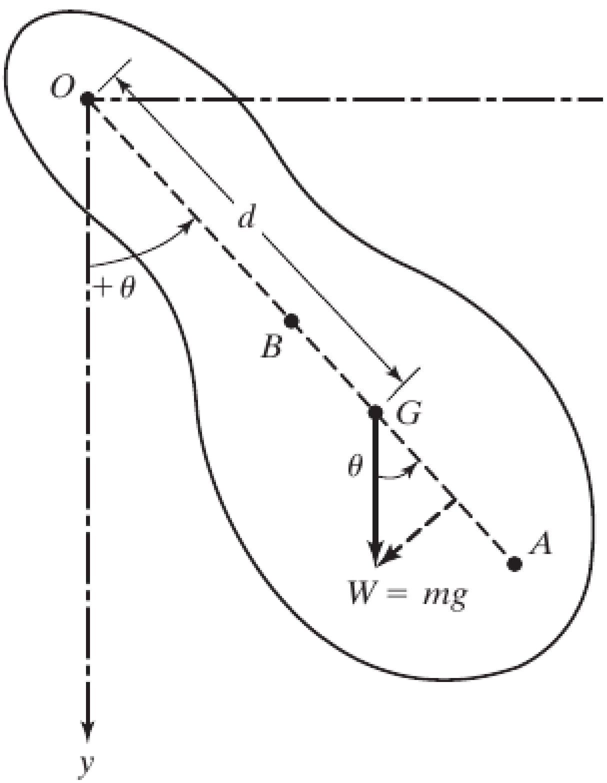
Centre of Percussion



$$\omega_n = \left(\frac{Wd}{J_0}\right)^{1/2} = \left(\frac{mgd}{J_0}\right)^{1/2}$$

Comparing this natural frequency with the natural frequency of a simple pendulum

$$\omega_n = (g/l)^{1/2}$$

Centre of Percussion

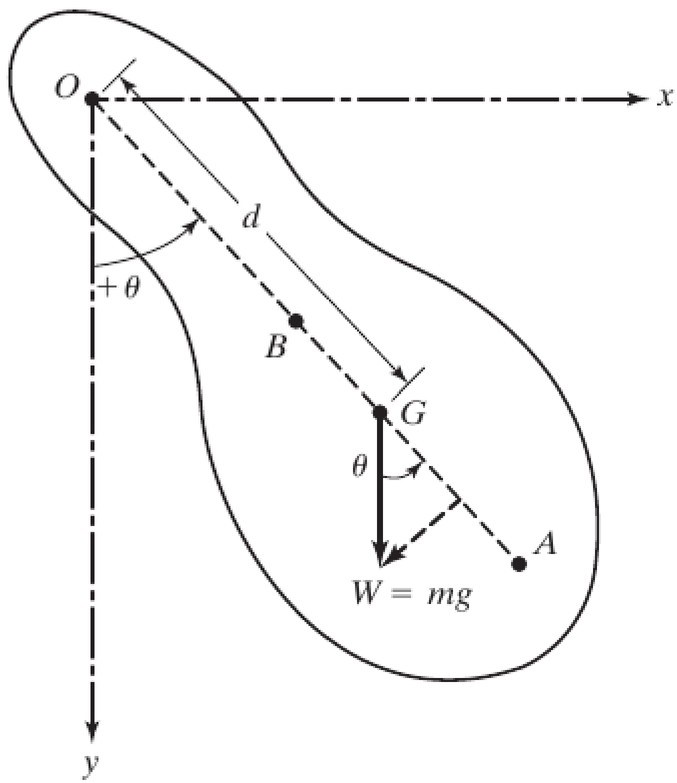
$$\omega_n = \left(\frac{mgd}{J_0}\right)^{1/2} \qquad \omega_n = (g/l)^{1/2}$$

we can find the length of the equivalent simple pendulum:

$$l = \frac{J_0}{md}$$

Replacing J_0 by mk_0^2, where k_0 is the radius of gyration of the body about O

Centre of Percussion

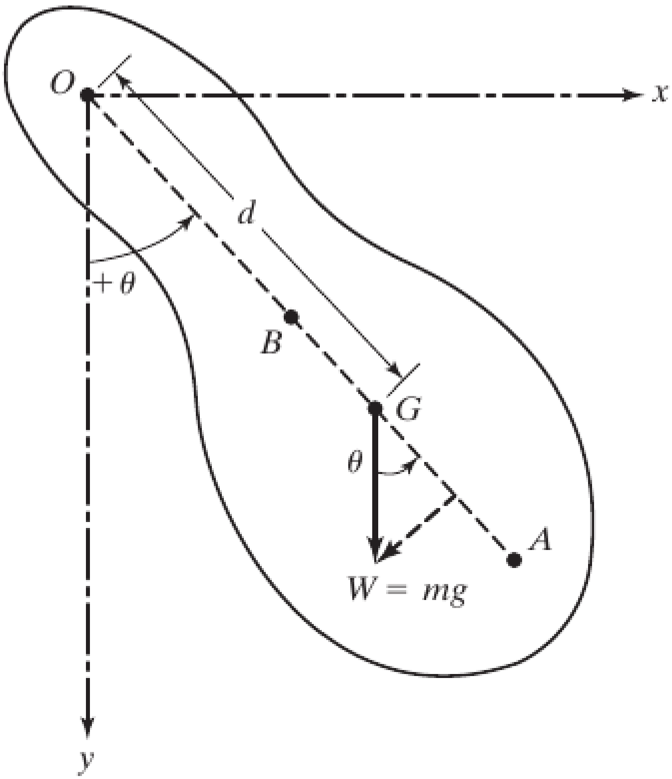


$$\omega_n = \left(\frac{gd}{k_0^2}\right)^{1/2} \quad l = \left(\frac{k_0^2}{d}\right)$$

$$k_0^2 = k_G^2 + d^2$$

$$l = \left(\frac{k_G^2}{d} + d\right)$$

Centre of Percussion



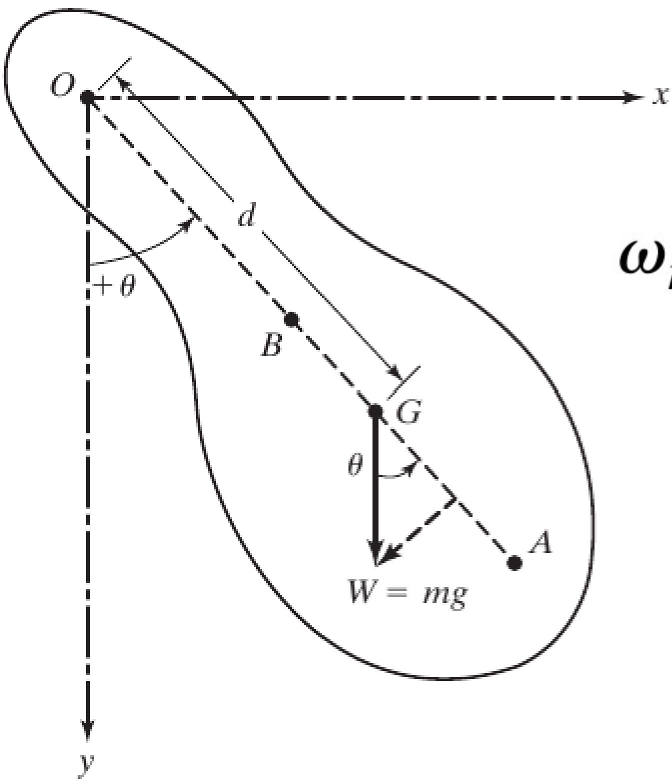
$$l = \left(\frac{k_G^2}{d} + d \right)$$

$$GA = \frac{k_G^2}{d}$$

$$l = GA + d = OA$$

$$\omega_n = \left\{ \frac{g}{(k_0^2/d)} \right\}^{1/2} = \left(\frac{g}{l} \right)^{1/2} = \left(\frac{g}{OA} \right)^{1/2}$$

Centre of Percussion



$$\omega_n = \left\{ \frac{g}{(k_0^2/d)} \right\}^{1/2} = \left(\frac{g}{l} \right)^{1/2} = \left(\frac{g}{OA} \right)^{1/2}$$

This equation shows that, no matter whether the body is pivoted from O or A, its natural frequency is the same. The point A is called the centre of percussion.

Centre of Percussion