Mechanical Vibrations

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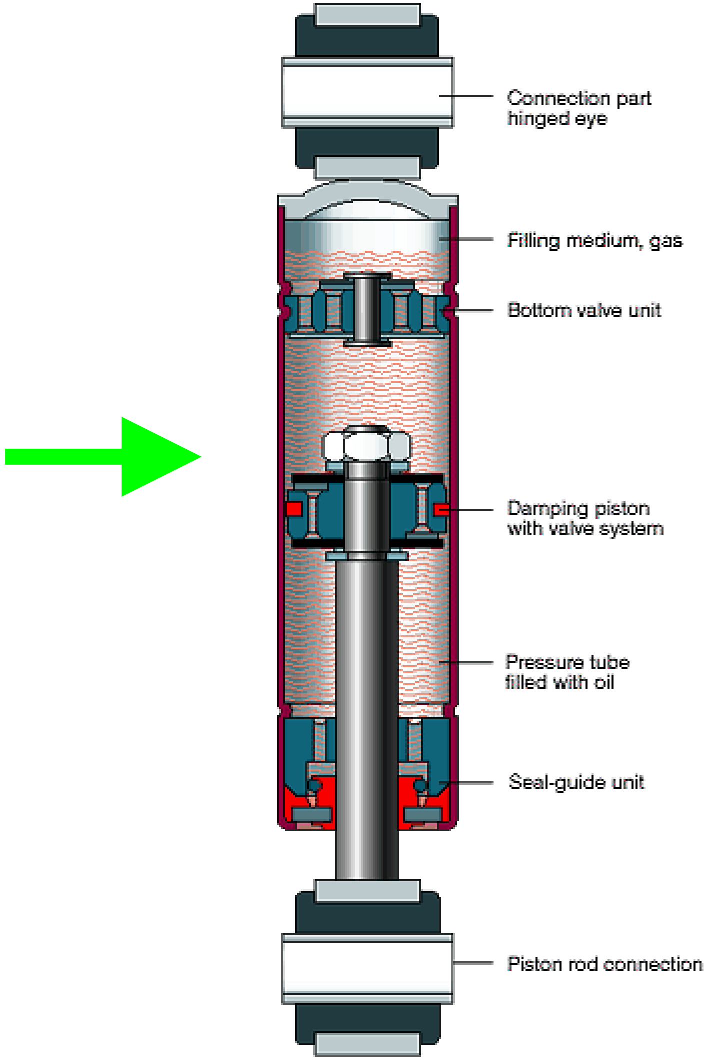
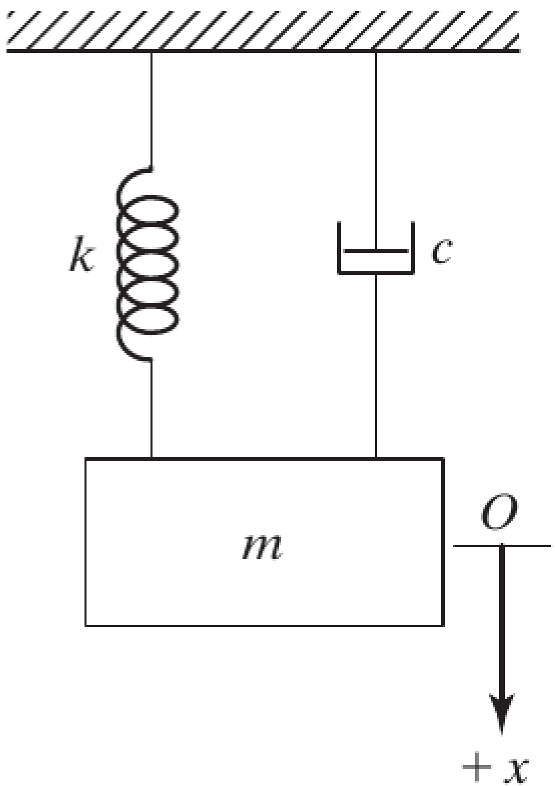
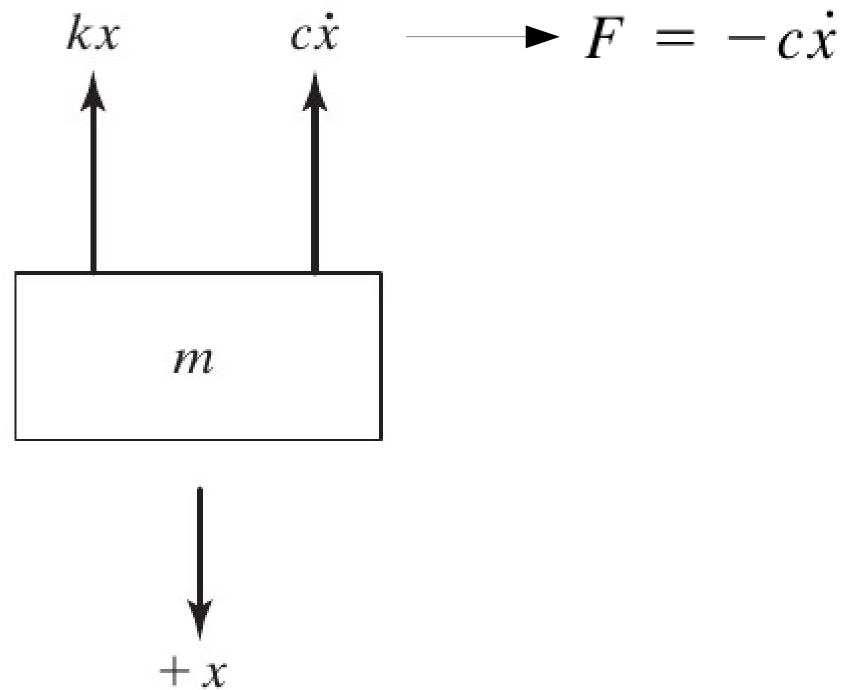
Viscous Damping



Example of Application

Free Vibration with Viscous Damping





$$m\ddot{x} = -c\dot{x} - kx \longrightarrow m\ddot{x} + c\dot{x} + kx = 0$$

Free Vibration with Viscous Damping

$$m\ddot{x} + c\dot{x} + kx = 0 \longrightarrow x(t) = Ce^{st}$$

$$ms^2 + cs + k = 0$$

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$x_1(t) = C_1 e^{s_1 t} \quad \text{and} \quad x_2(t) = C_2 e^{s_2 t}$$

Critical Damping and Damping Ratio

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$$

Critical Damping

$$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2m\omega_n$$

Damping Ratio

$$\zeta = c/c_c \longrightarrow \frac{c}{2m} = \frac{c}{c_c} \cdot \frac{c_c}{2m} = \zeta\omega_n$$

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

Underdamped Case

$$(\zeta < 1 \text{ or } c < c_c \text{ or } c/2m < \sqrt{k/m})$$

$$s_1 = (-\zeta + i\sqrt{1 - \zeta^2})\omega_n$$
$$s_2 = (-\zeta - i\sqrt{1 - \zeta^2})\omega_n$$

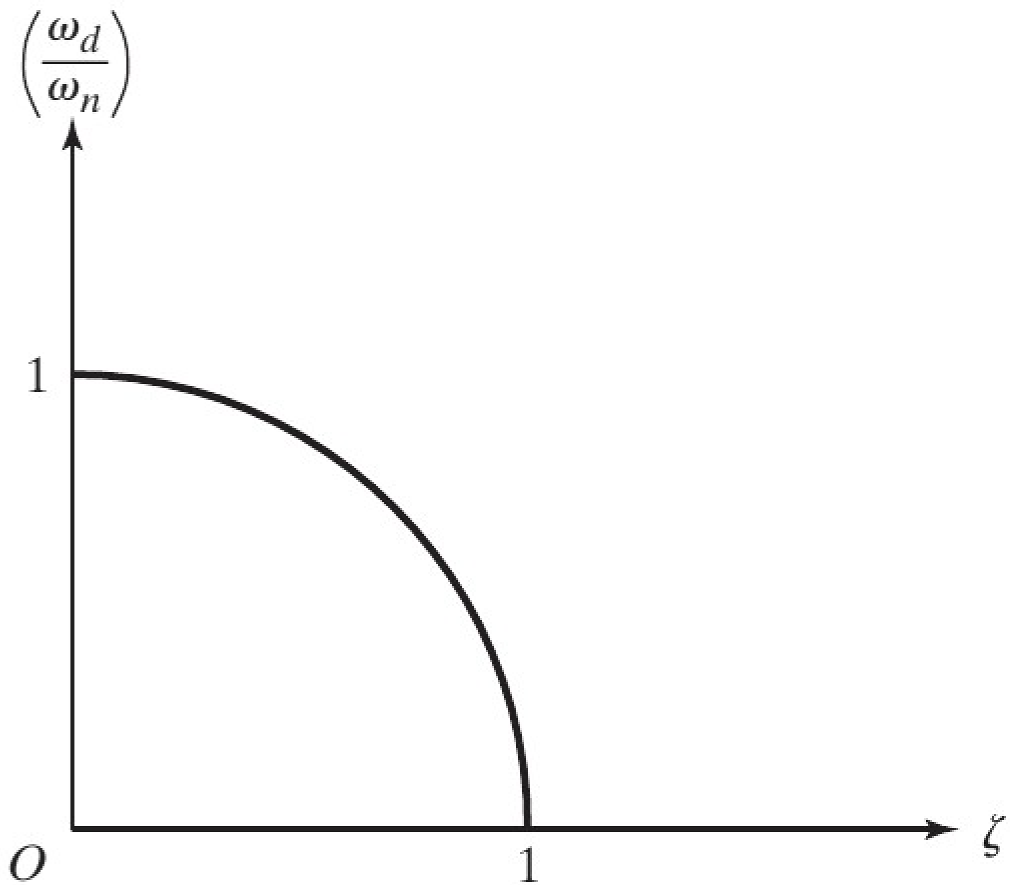
$$x(t) = C_1 e^{(-\zeta + i\sqrt{1-\zeta^2})\omega_n t} + C_2 e^{(-\zeta - i\sqrt{1-\zeta^2})\omega_n t}$$
$$= e^{-\zeta\omega_n t}\left\{C_1 e^{i\sqrt{1-\zeta^2}\omega_n t} + C_2 e^{-i\sqrt{1-\zeta^2}\omega_n t}\right\}$$

Underdamped Case

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos \sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n} \sin\sqrt{1-\zeta^2}\,\omega_n t\right\}$$

Damped Natural Frequency $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$

Damped Natural Frequency



$$\omega_d = \sqrt{1 - \zeta^2}\,\omega_n$$

Underdamped Case

$$x(t) = X_0 e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi_0\right)$$

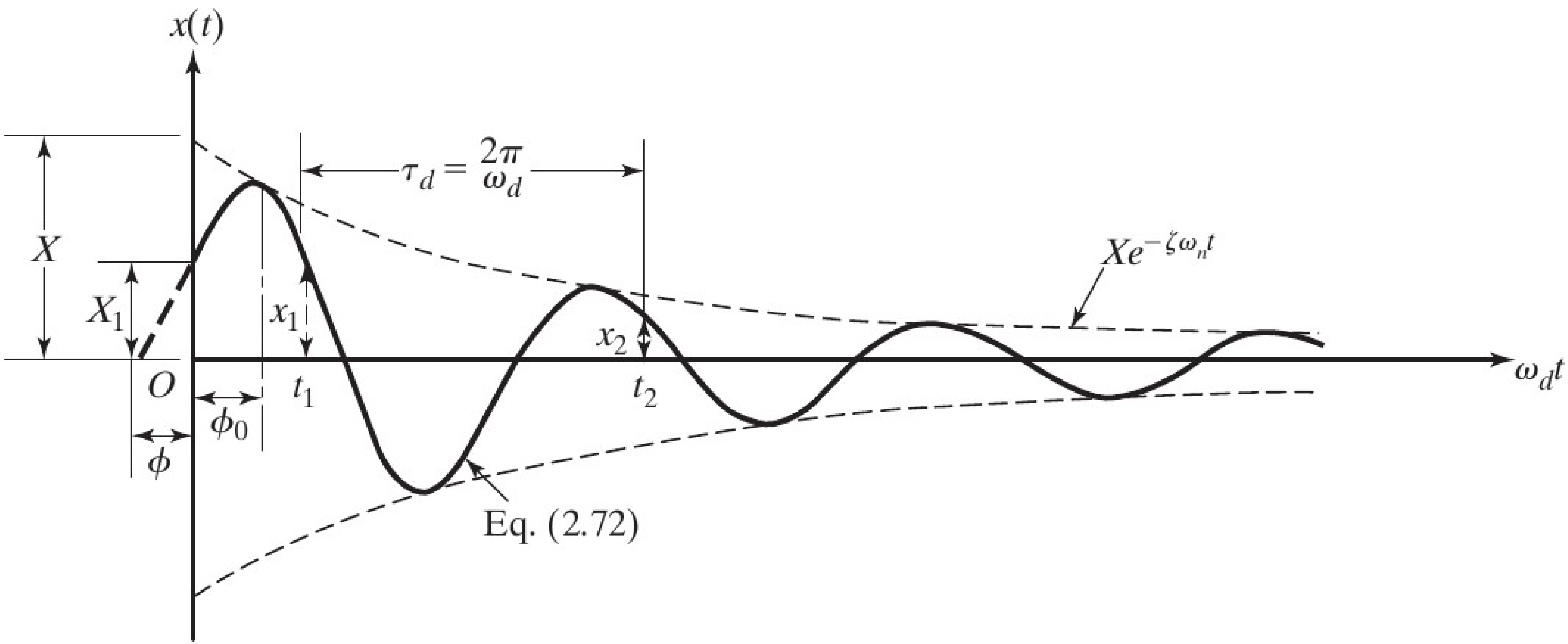
$$x(t) = X e^{-\zeta\omega_n t} \cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \phi\right)$$

$$X = X_0 = \sqrt{(C_1')^2 + (C_2')^2} = \frac{\sqrt{x_0^2\omega_n^2 + \dot{x}_0^2 + 2x_0\dot{x}_0\zeta\omega_n}}{\sqrt{1-\zeta^2}\,\omega_n}$$

$$\phi_0 = \tan^{-1}\left(\frac{C_1'}{C_2'}\right) = \tan^{-1}\left(\frac{x_0\omega_n\sqrt{1-\zeta^2}}{\dot{x}_0 + \zeta\omega_n x_0}\right)$$

$$\phi = \tan^{-1}\left(\frac{C_2'}{C_1'}\right) = \tan^{-1}\left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{x_0\omega_n\sqrt{1-\zeta^2}}\right)$$

Underdamped Case



Critically damped system

$(\zeta = 1 \text{ or } c = c_c \text{ or } c/2m = \sqrt{k/m}).$

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$= 0$

$$s_1 = s_2 = -\frac{c_c}{2m} = -\omega_n$$

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t}$$

$$x(t) = [x_0 + (\dot{x}_0 + \omega_n x_0)t]e^{-\omega_n t}$$

Overdamped system

$(\zeta > 1 \text{ or } c > c_c \text{ or } c/2m > \sqrt{k/m})$.

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

> 0

$$s_1 = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n < 0$$

$$s_2 = \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n < 0$$

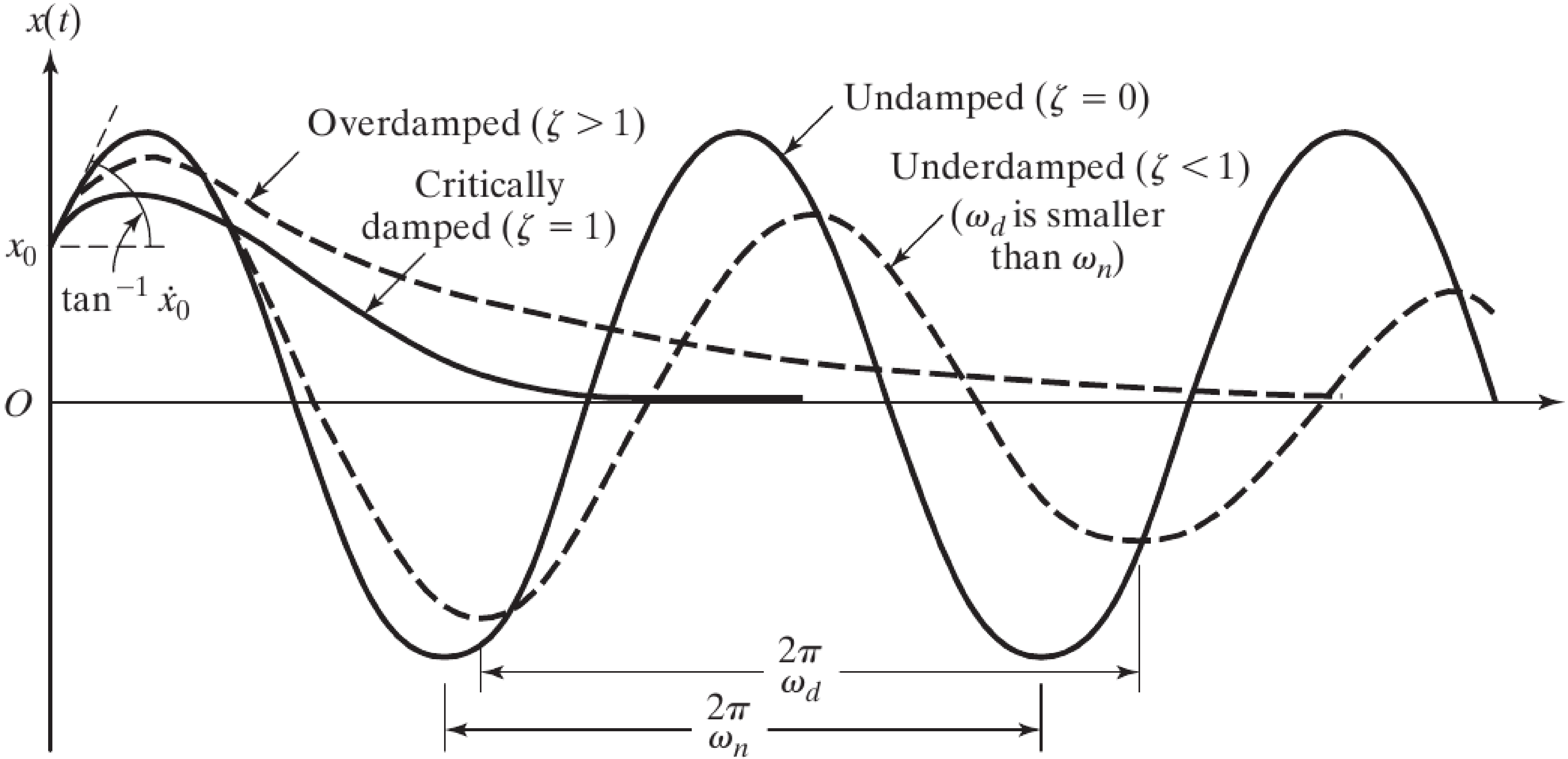
Overdamped system

$$x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

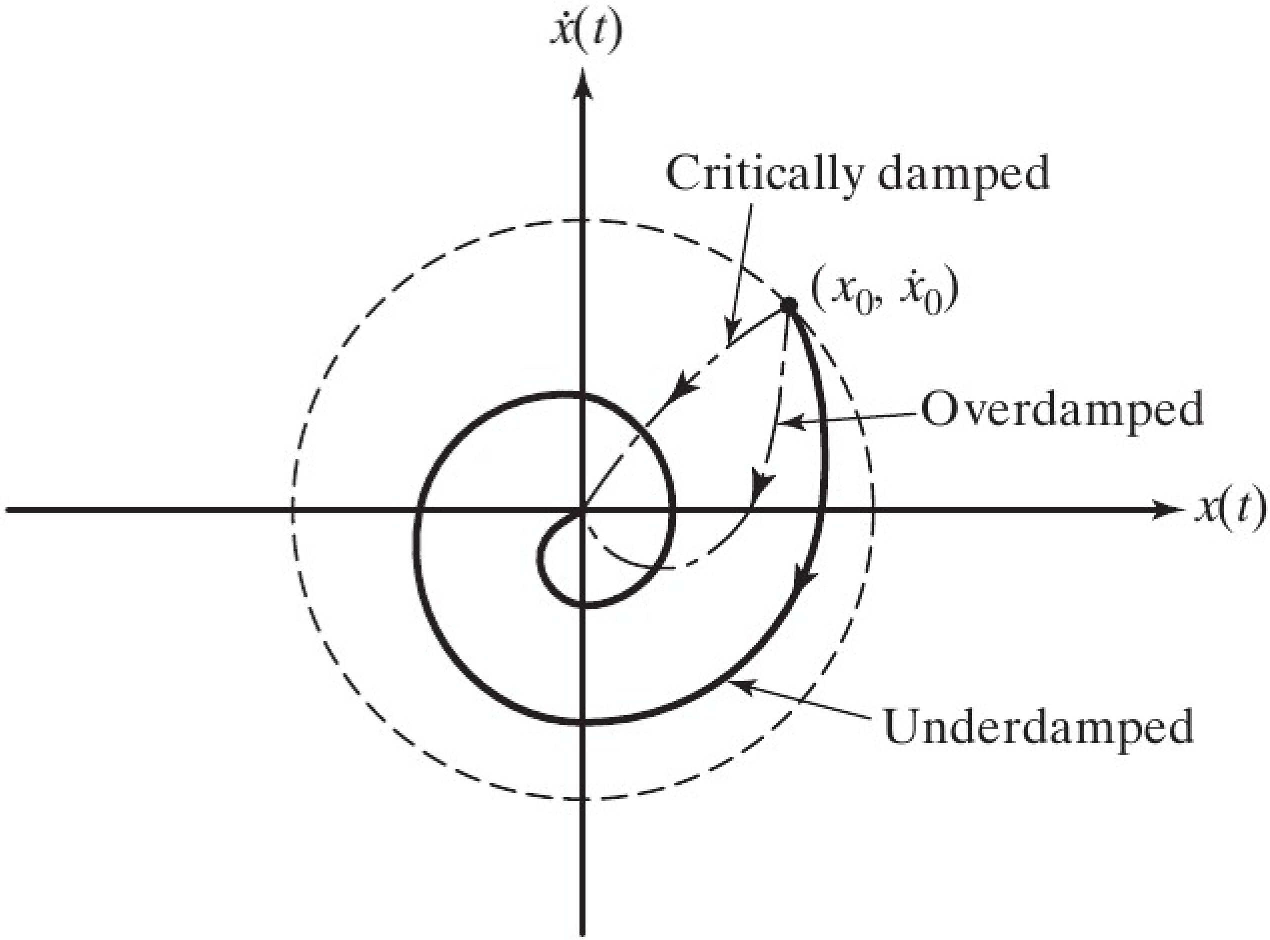
$$C_1 = \frac{x_0 \omega_n (\zeta + \sqrt{\zeta^2 - 1}) + \dot{x}_0}{2\omega_n \sqrt{\zeta^2 - 1}}$$

$$C_2 = \frac{-x_0 \omega_n (\zeta - \sqrt{\zeta^2 - 1}) - \dot{x}_0}{2\omega_n \sqrt{\zeta^2 - 1}}$$

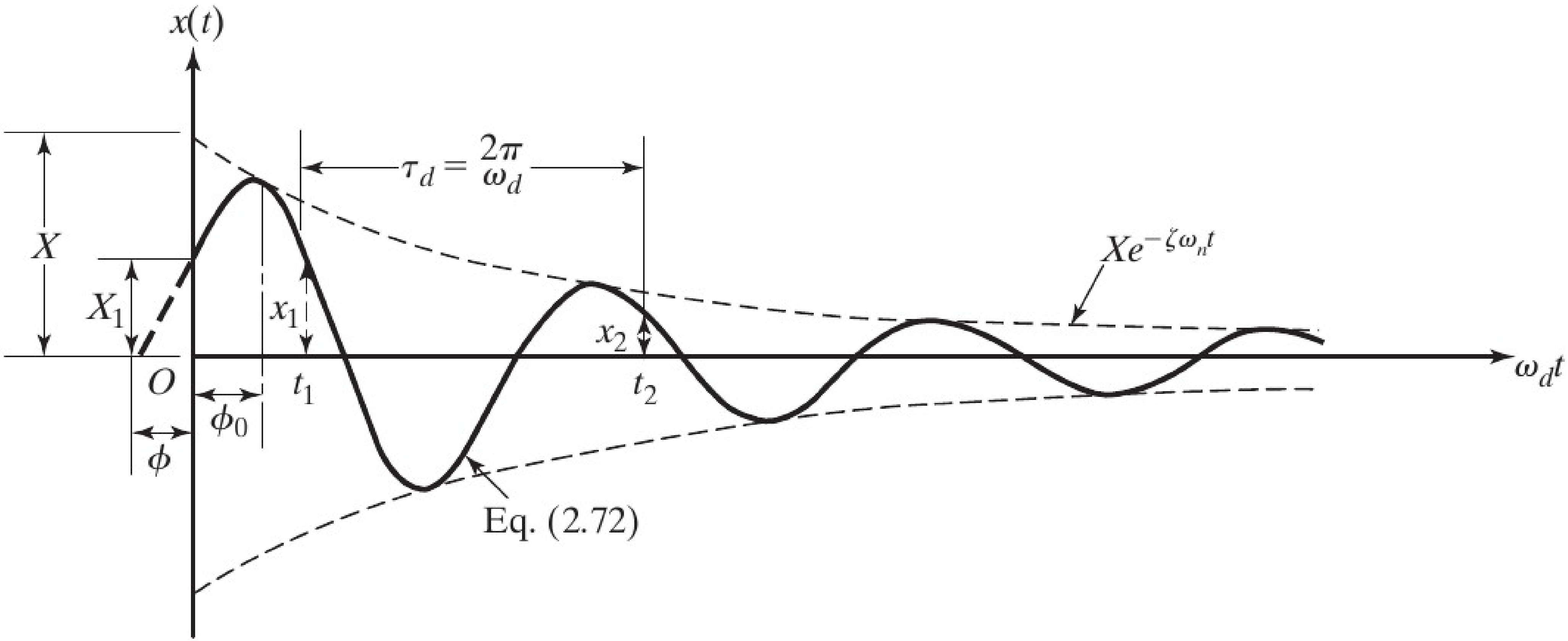
Damped Vibration



Damped Vibration

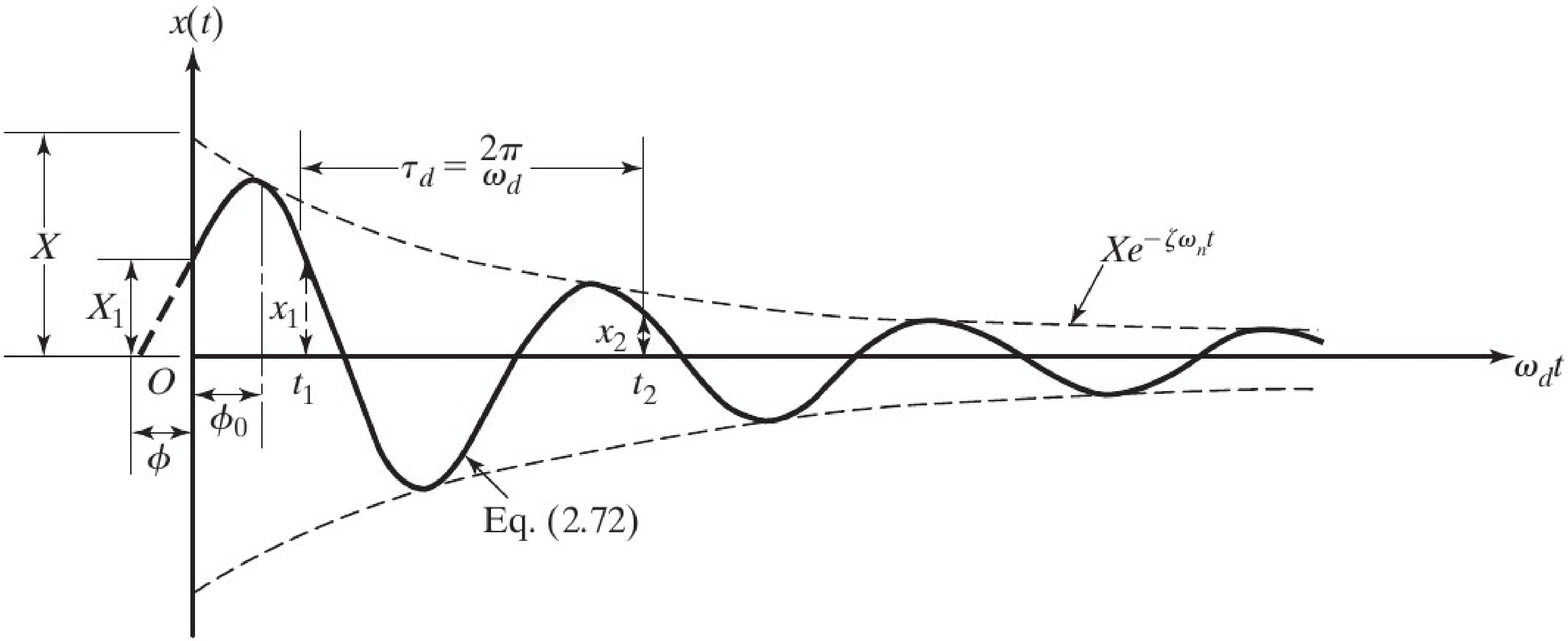


Logarithmic Decrement



$$\frac{x_1}{x_2} = \frac{X_0 e^{-\zeta\omega_n t_1}\cos(\omega_d t_1 - \phi_0)}{X_0 e^{-\zeta\omega_n t_2}\cos(\omega_d t_2 - \phi_0)}$$

Logarithmic Decrement



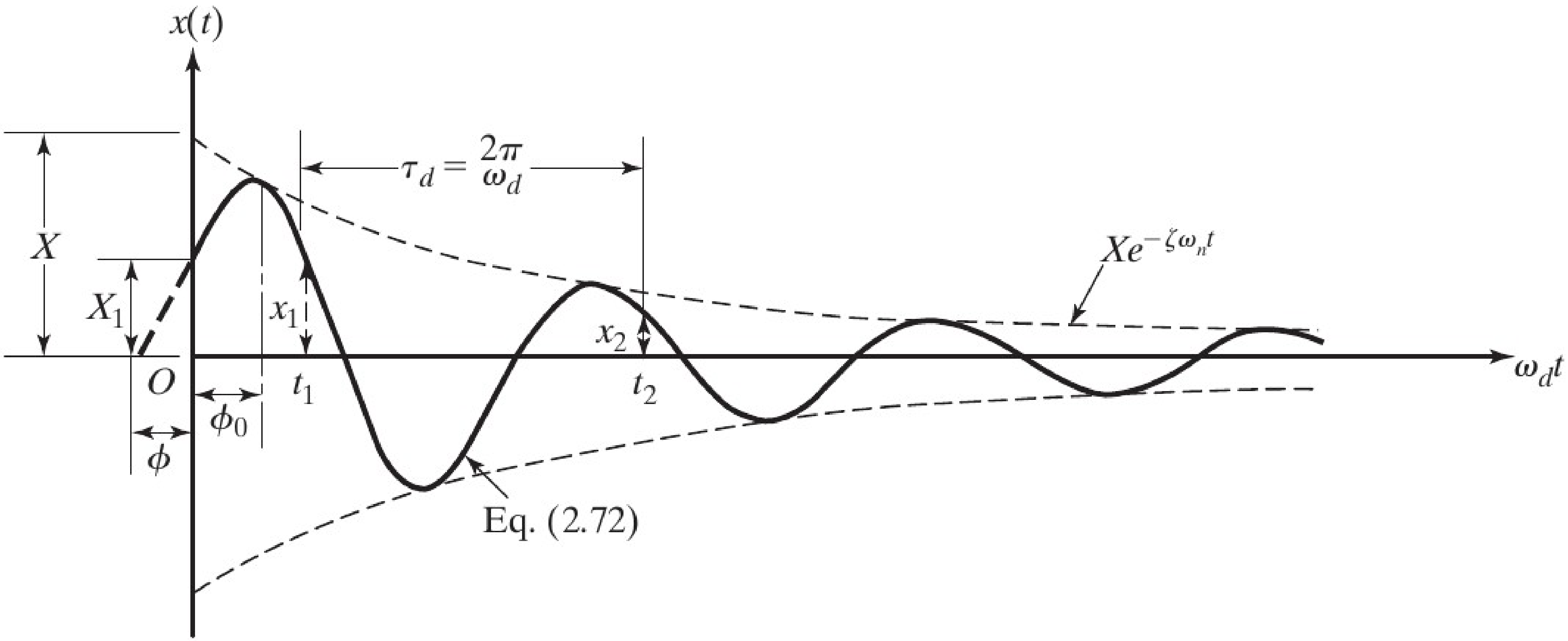
$$\frac{x_1}{x_2} = \frac{X_0 e^{-\zeta\omega_n t_1}\cos(\omega_d t_1 - \phi_0)}{X_0 e^{-\zeta\omega_n t_2}\cos(\omega_d t_2 - \phi_0)}$$

$$t_2 = t_1 + \tau_d$$

$$\tau_d = 2\pi/\omega_d$$

$$\cos(\omega_d t_2 - \phi_0) = \cos(2\pi + \omega_d t_1 - \phi_0) = \cos(\omega_d t_1 - \phi_0)$$

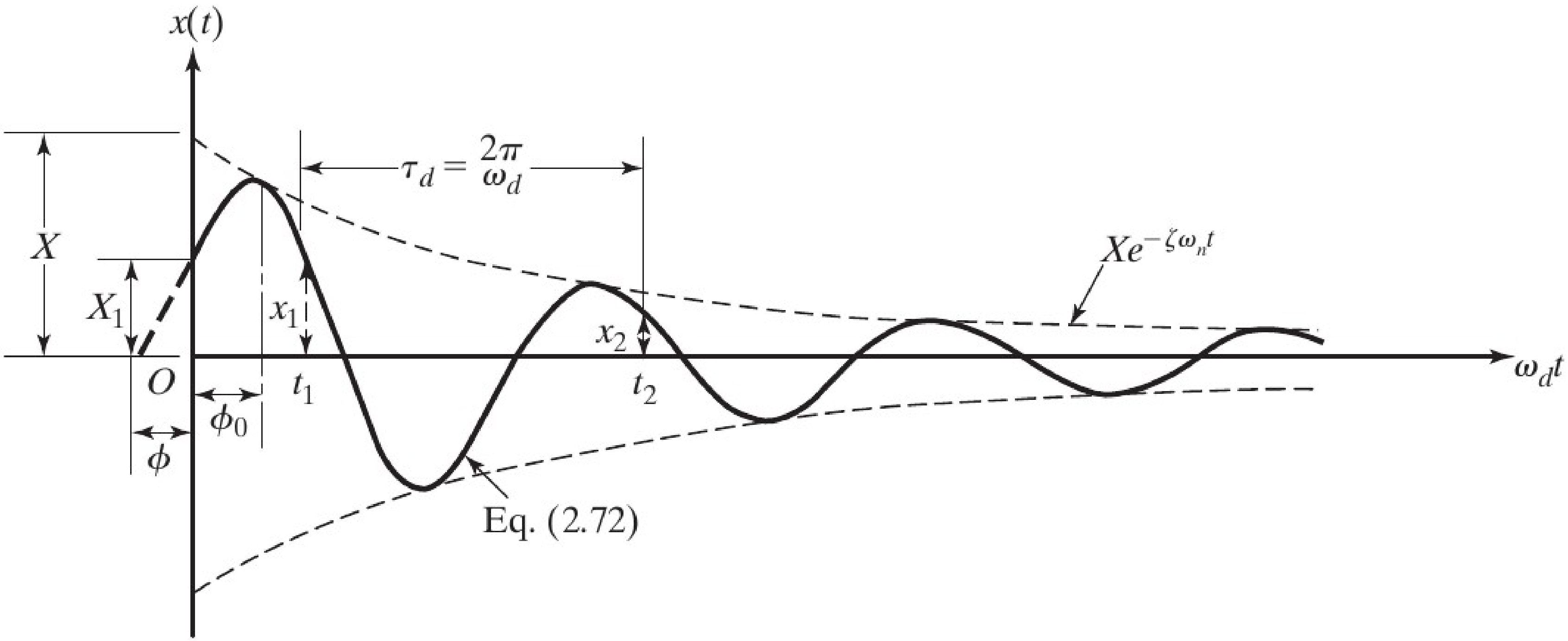
Logarithmic Decrement



$$\frac{x_1}{x_2} = \frac{X_0 e^{-\zeta\omega_n t_1}\cos(\omega_d t_1 - \phi_0)}{X_0 e^{-\zeta\omega_n t_2}\cos(\omega_d t_2 - \phi_0)} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n(t_1+\tau_d)}} = e^{\zeta\omega_n\tau_d}$$

$$\delta = \ln\frac{x_1}{x_2} = \zeta\omega_n\tau_d = \zeta\omega_n\frac{2\pi}{\sqrt{1-\zeta^2}\omega_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d}\cdot\frac{c}{2m}$$

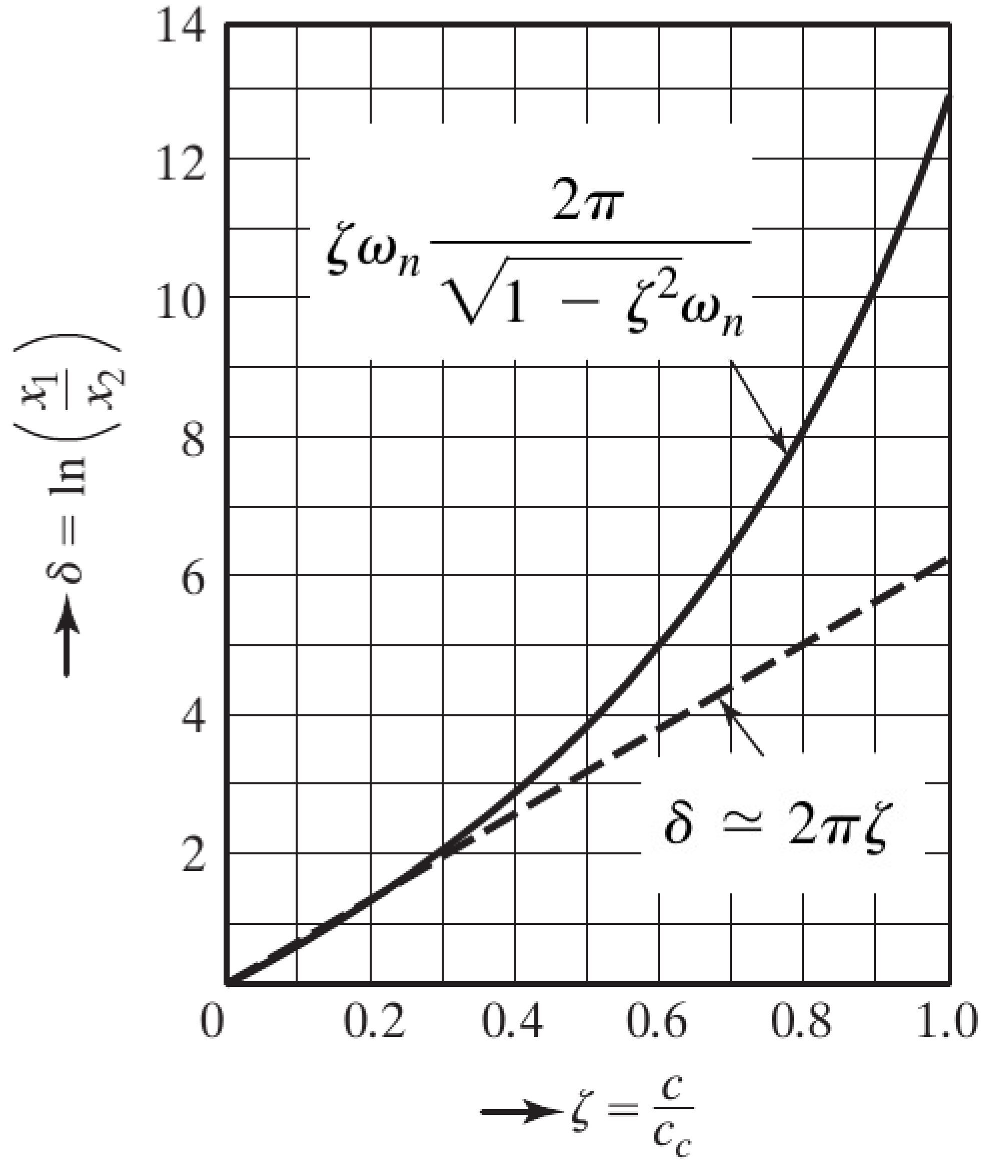
Logarithmic Decrement



$$\delta = \ln\frac{x_1}{x_2} = \zeta\omega_n\tau_d = \zeta\omega_n\frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d}\cdot\frac{c}{2m}$$

$$\delta \simeq 2\pi\zeta \qquad \text{if} \qquad \zeta \ll 1$$

Logarithmic Decrement



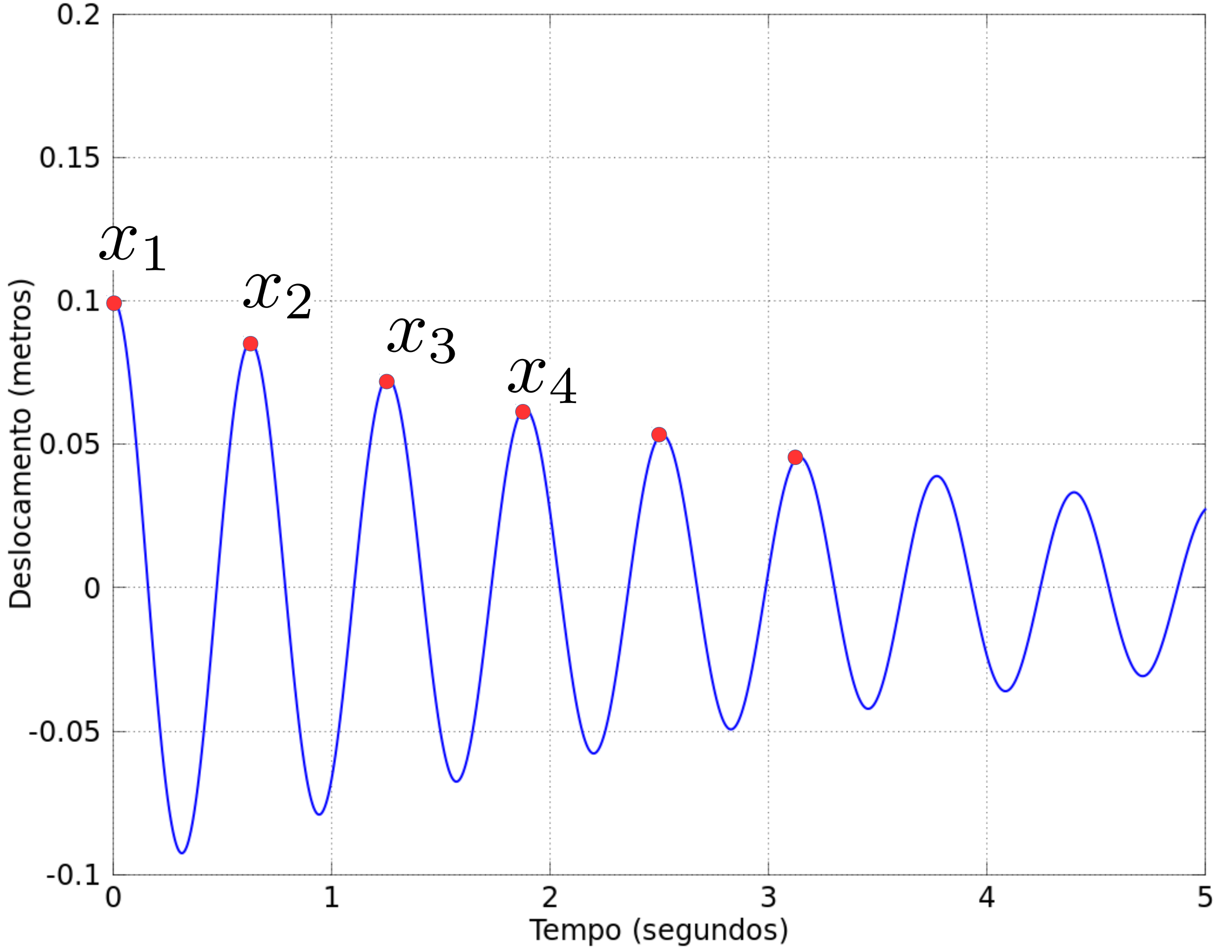
Logarithmic Decrement

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$

If damping is small

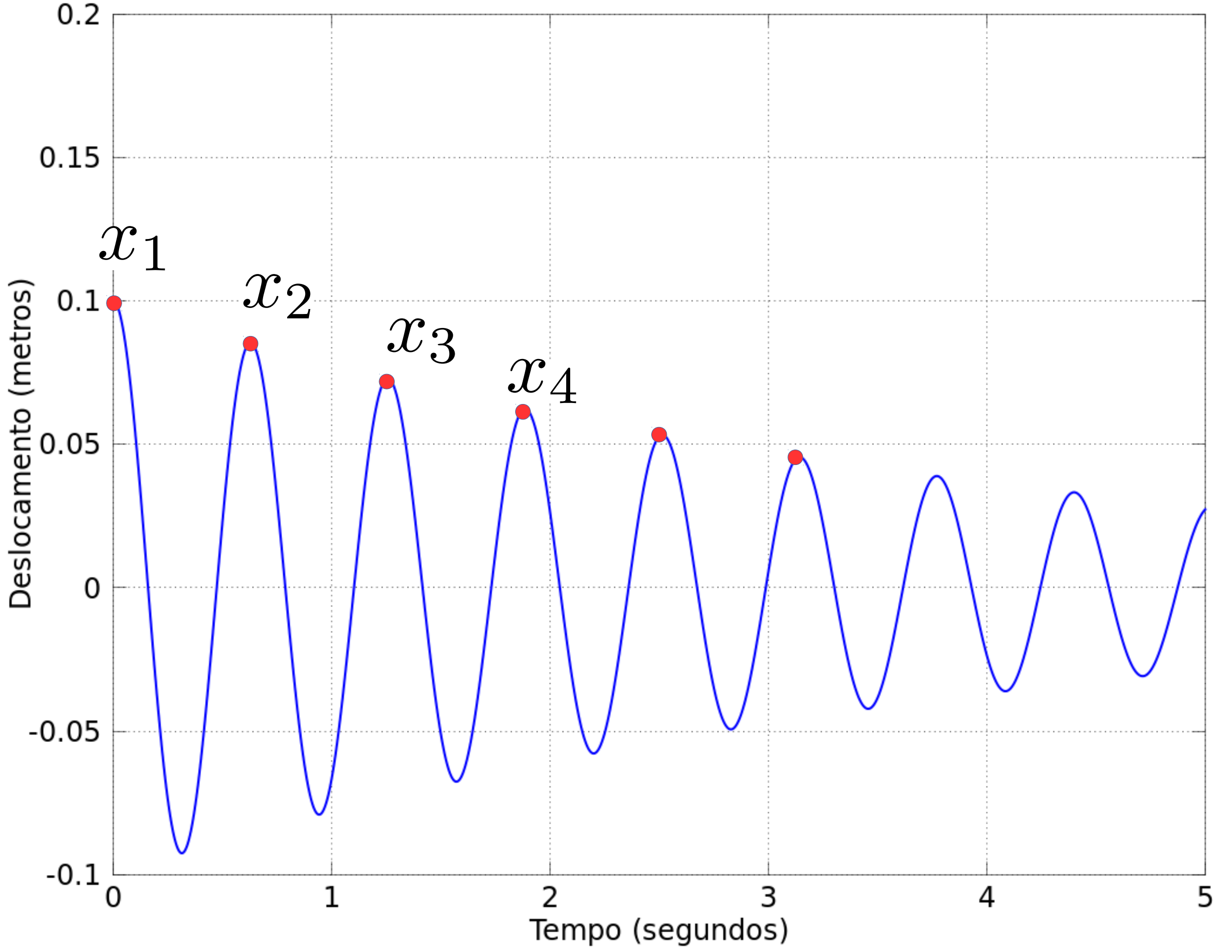
$$\zeta \simeq \frac{\delta}{2\pi}$$

Generalization



$$\frac{x_1}{x_{m+1}} = \frac{x_1}{x_2}\frac{x_2}{x_3}\frac{x_3}{x_4}\cdots\frac{x_m}{x_{m+1}}$$

Generalization

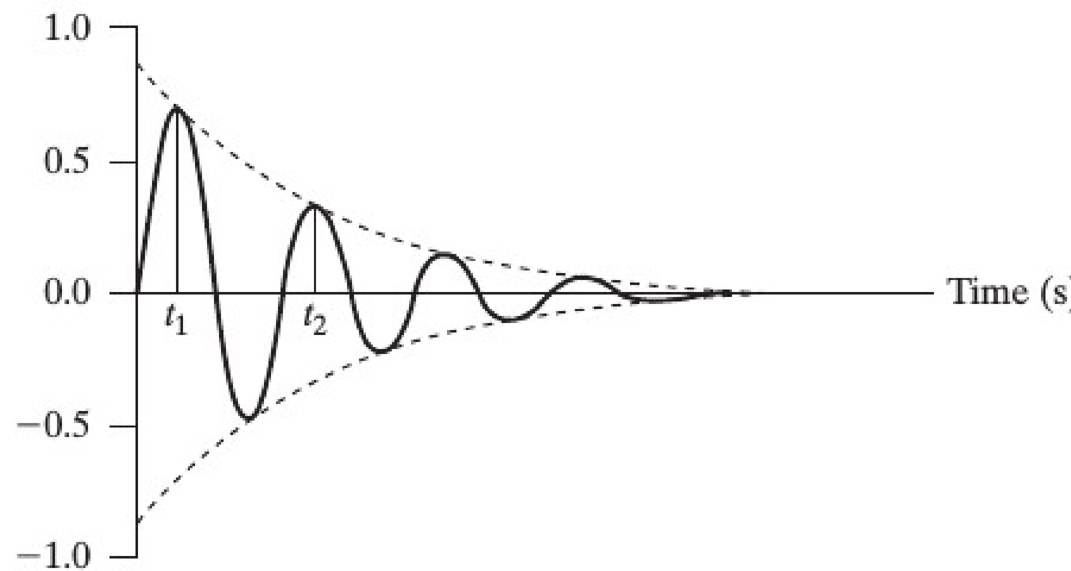


$$\frac{x_1}{x_{m+1}} = e^{m(\zeta \omega_n \tau_d)}$$

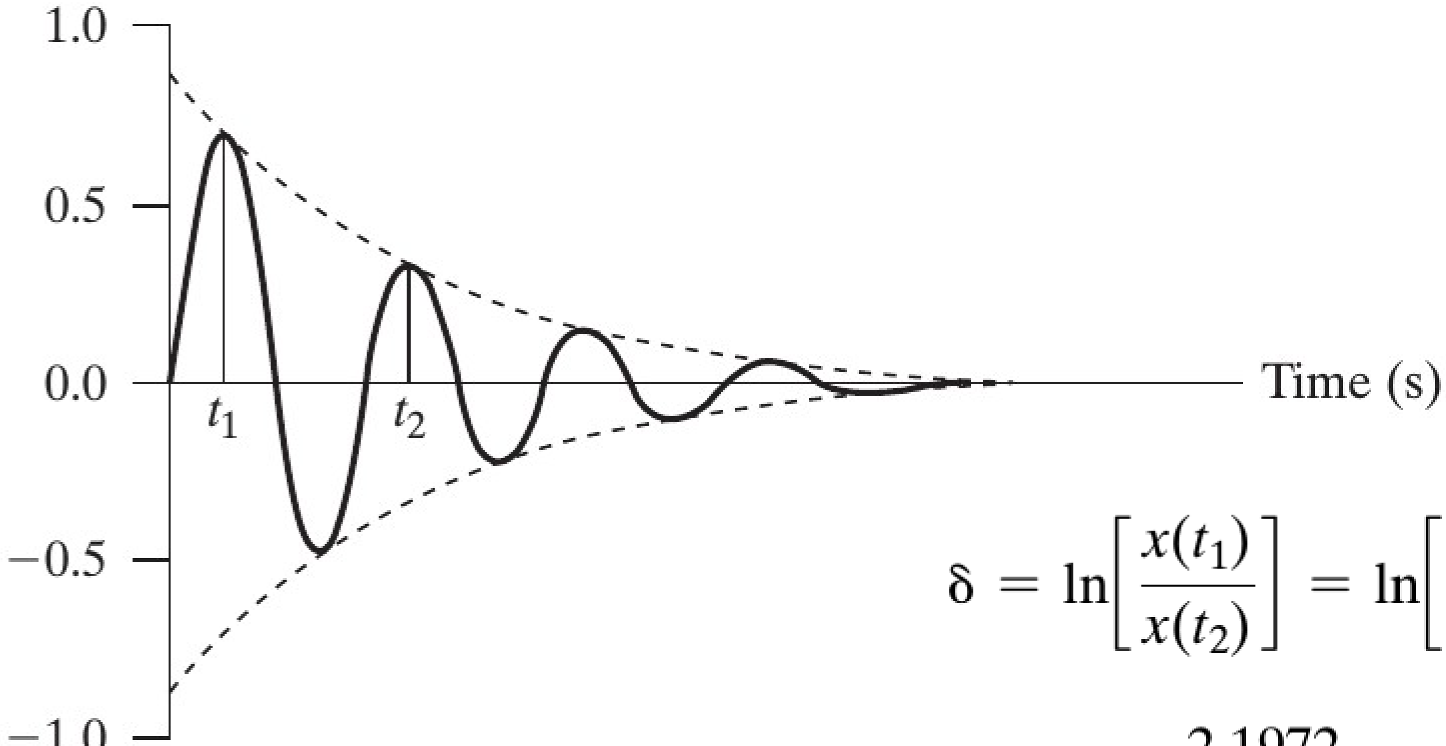
$$\delta = \frac{1}{m} \ln \frac{x_1}{x_{m+1}}$$

Example

The free response of the damped single degree of freedom with mass of 2 kg is recorded to be of the form given in figure below. A static deflection test is performed and the stiffens is determined to be 1.5 x 10^3 N/m. The displacements at time t_1 and t_2 are measured to be 9 and 1 mm, respectively. Calculate the damping coefficient.



Example



$$\delta = \ln\left[\frac{x(t_1)}{x(t_2)}\right] = \ln\left[\frac{9\text{ mm}}{1\text{ mm}}\right] = 2.1972$$

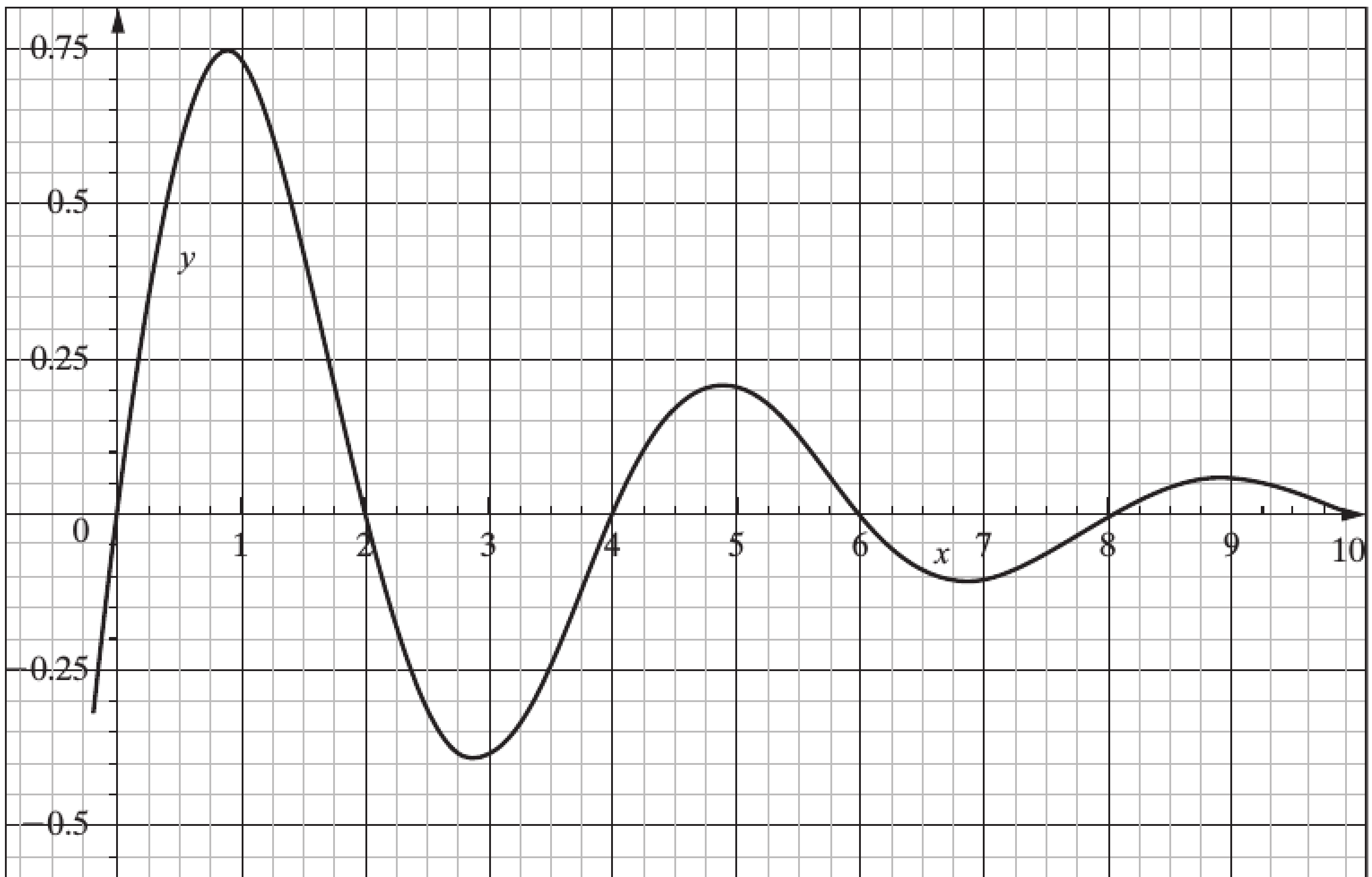
$$\zeta = \frac{2.1972}{\sqrt{4\pi^2 + 2.1972^2}} = 0.33 \quad \text{or} \quad 33\%$$

$$c_{cr} = 2\sqrt{km} = 2\sqrt{(1.5 \times 10^3\text{ N/m})(2\text{ kg})} = 1.095 \times 10^2\text{ kg/s}$$

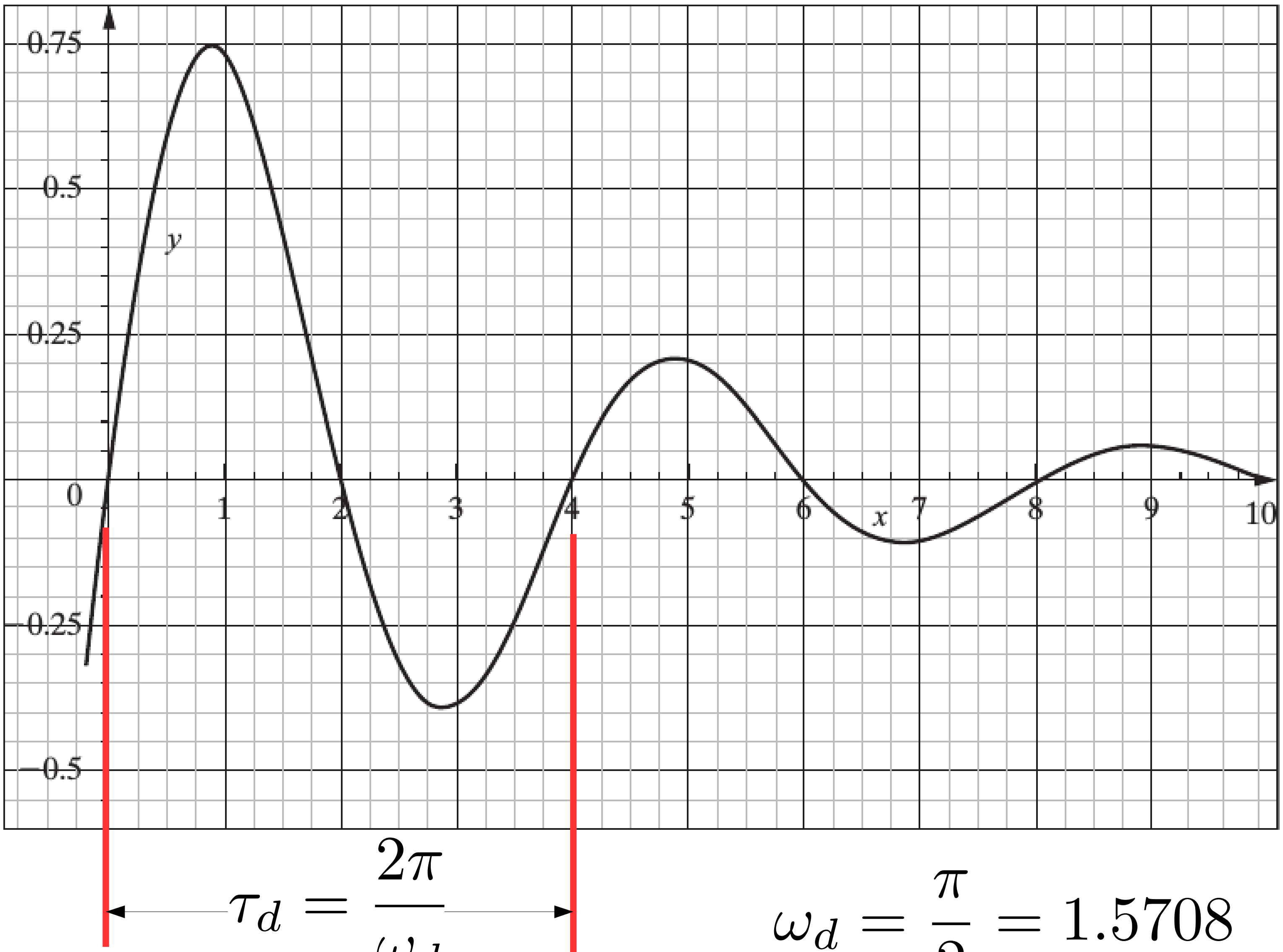
$$c = c_{cr}\zeta = \left(1.095 \times 10^2\right)(0.33) = 36.15\text{ kg/s}$$

Another Example

Determine the natural frequency, damping ratio and damped natural frequency



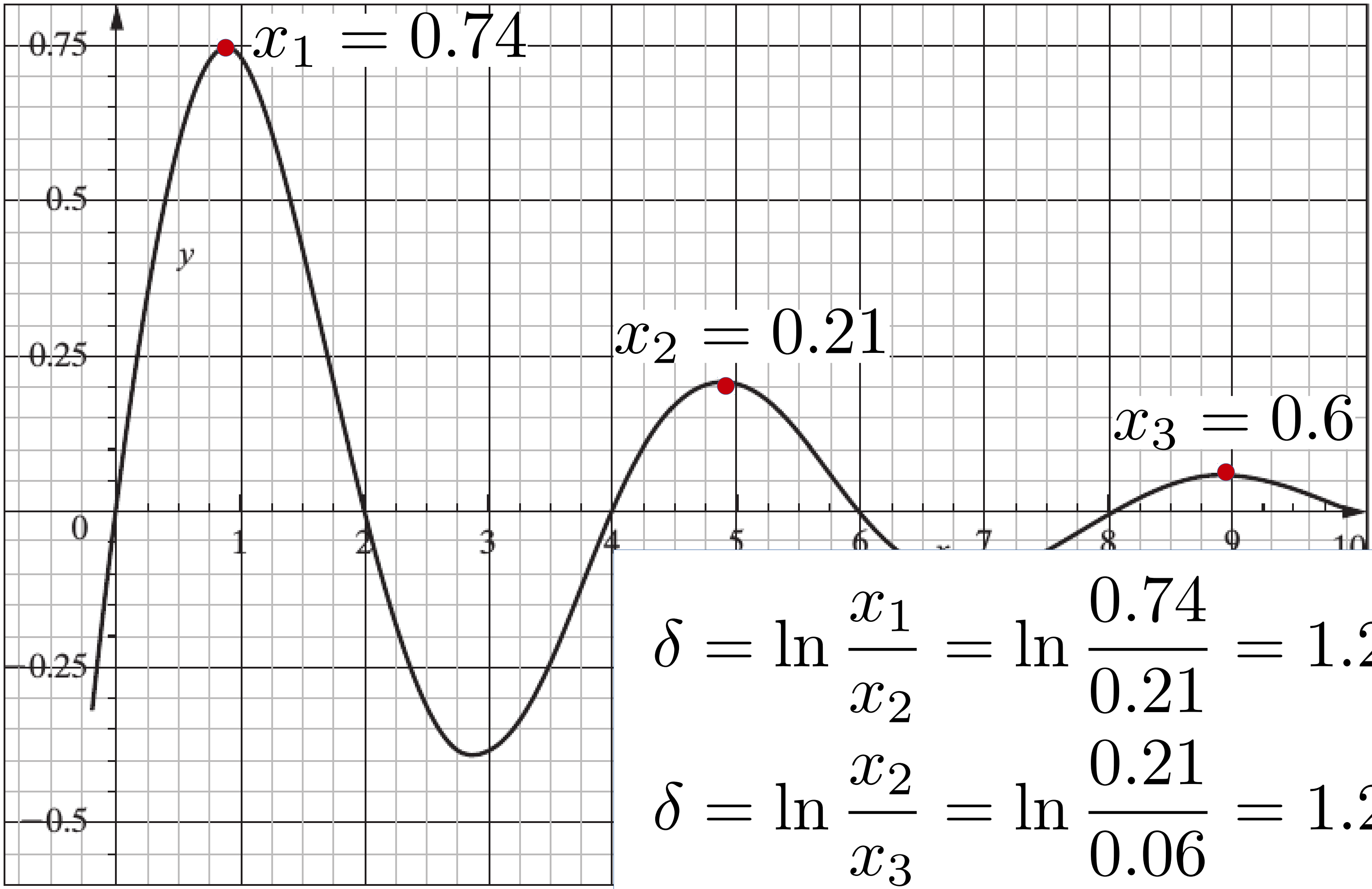
Another Example



$$\tau_d = \frac{2\pi}{\omega_d}$$

$$\omega_d = \frac{\pi}{2} = 1.5708$$

Another Example

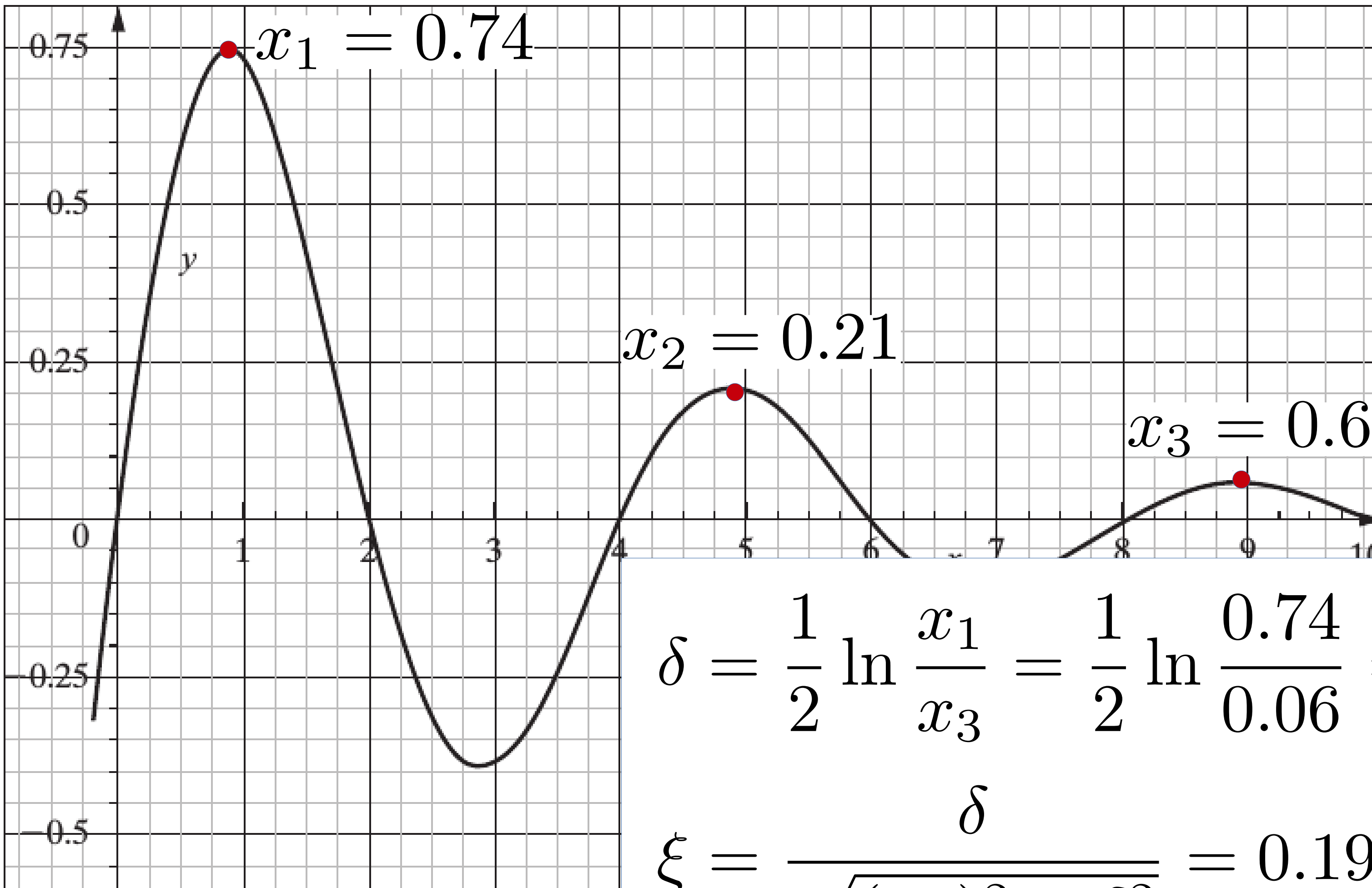


$$\delta = \ln \frac{x_1}{x_2} = \ln \frac{0.74}{0.21} = 1.2595$$

$$\delta = \ln \frac{x_2}{x_3} = \ln \frac{0.21}{0.06} = 1.2528$$

$$\delta = \frac{1}{2} \ln \frac{x_1}{x_3} = \frac{1}{2} \ln \frac{0.74}{0.06} = 1.2562$$

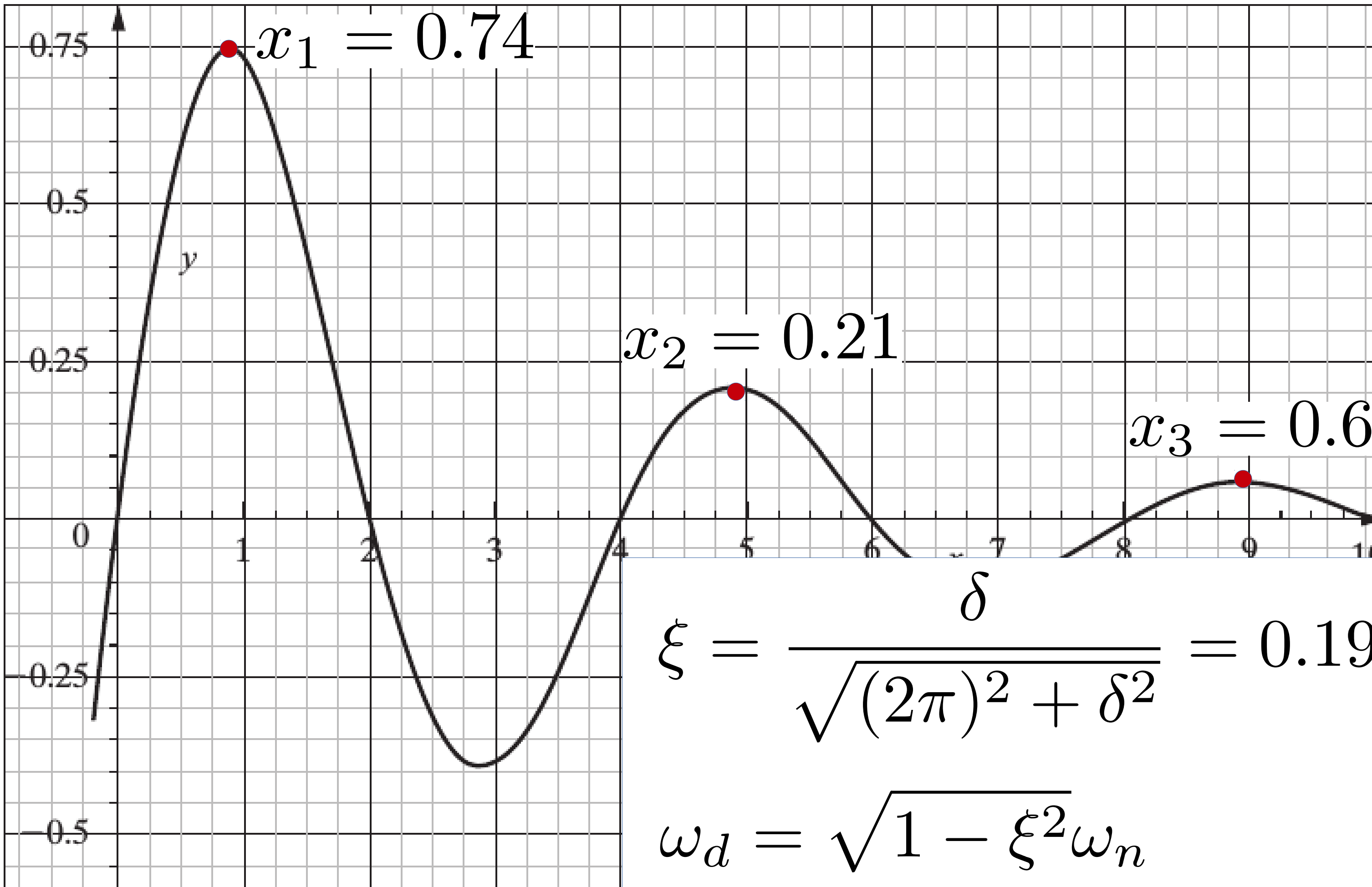
Another Example



$$\delta = \frac{1}{2} \ln \frac{x_1}{x_3} = \frac{1}{2} \ln \frac{0.74}{0.06} = 1.2562$$

$$\xi = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = 0.19604$$

Another Example



$$\xi = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = 0.19604$$

$$\omega_d = \sqrt{1 - \xi^2}\omega_n$$

$$\omega_n = \frac{\omega_d}{\sqrt{1 - xi^2}} = 1.6336$$