Mechanical Vibrations

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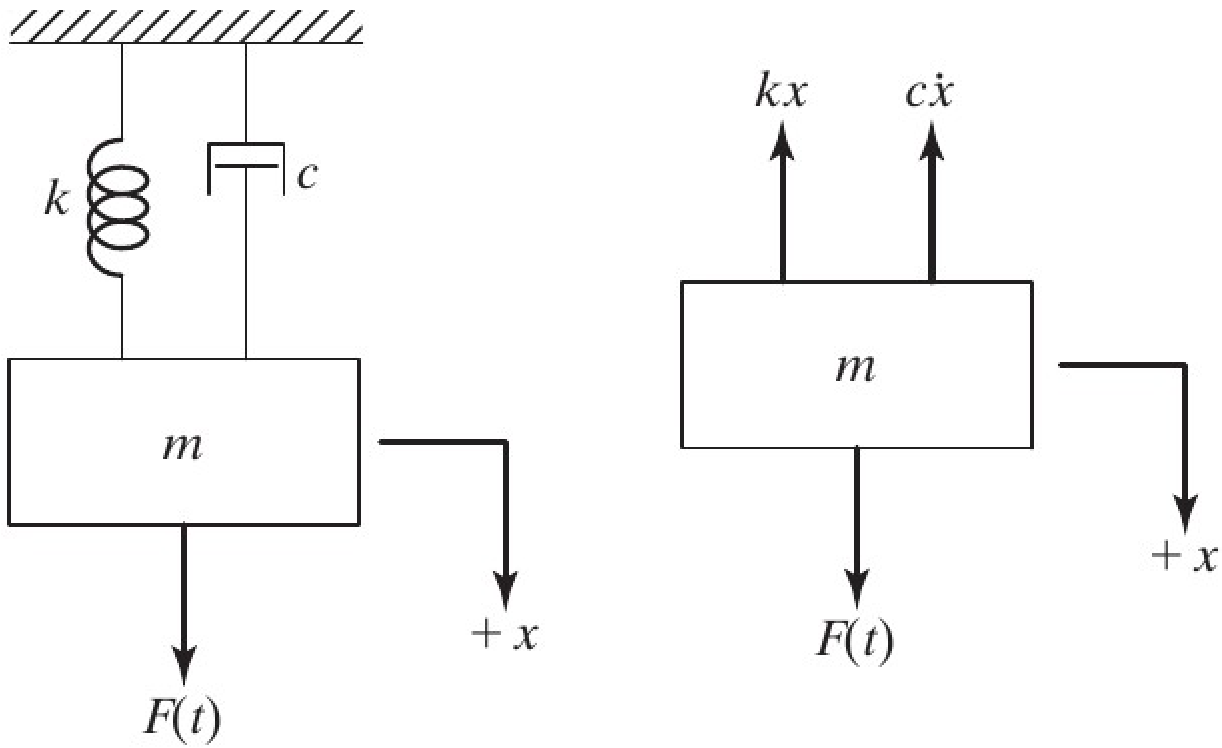
Harmonically Excited Vibration



$$m\ddot{x} + c\dot{x} + kx = F(t)$$

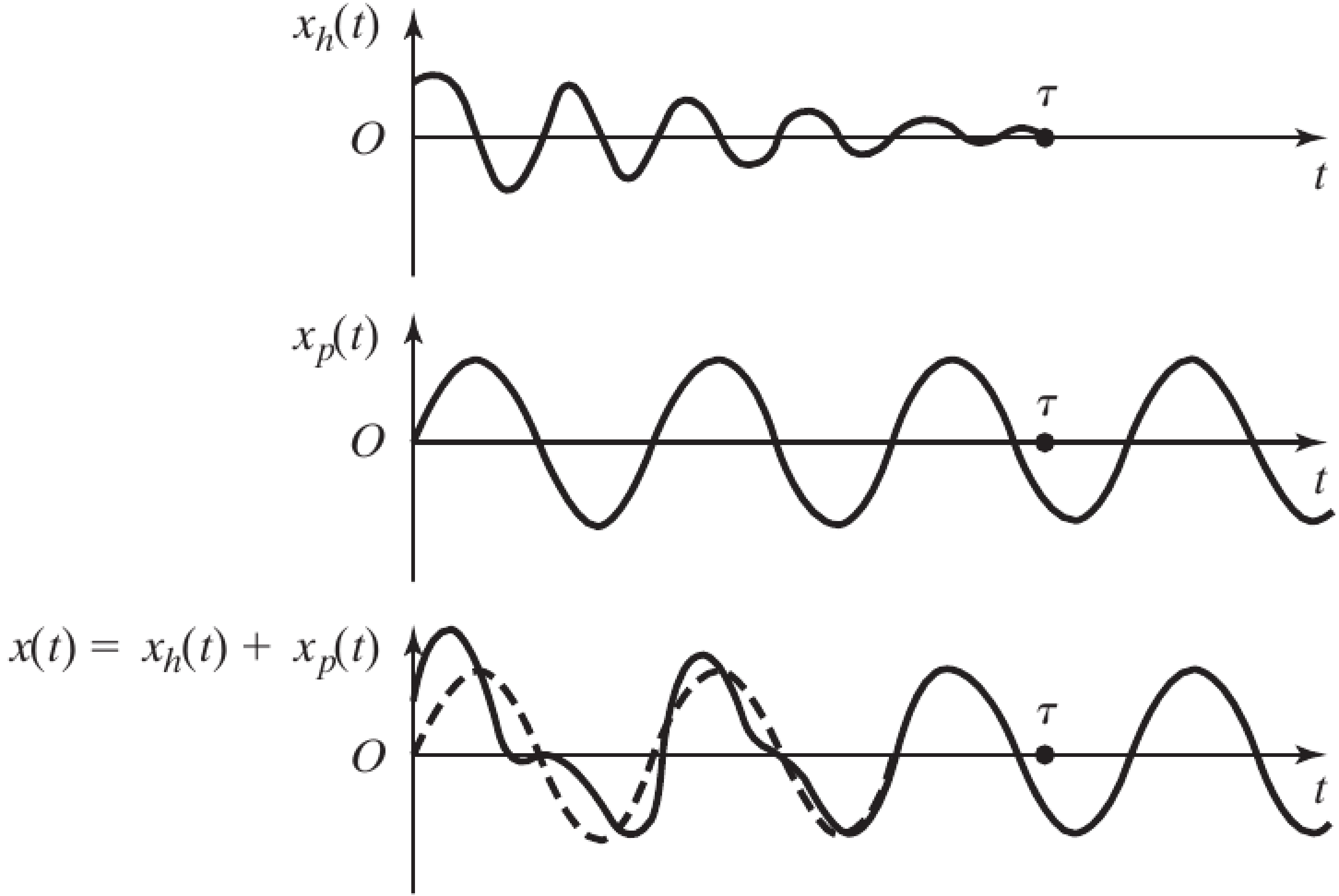
Harmonically Excited Vibration

$$m\ddot{x} + c\dot{x} + kx = f$$

The solution of nonhomogeneous ordinary differential equation can be written as a sum of the solutions for the homogeneous and particular ODE

$$x(t) = x_h(t) + x_p(t)$$

Harmonically Excited Vibration



Response of an Undamped System Under Harmonic Force

$$m\ddot{x} + kx = F_0 \cos \omega t$$

The homogeneous solution of this equation is given by

$$x_h(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t$$

and the particular solution

$$x_p(t) = X \cos \omega t$$

Particular Solution

$$x_p(t) = X \cos \omega t$$

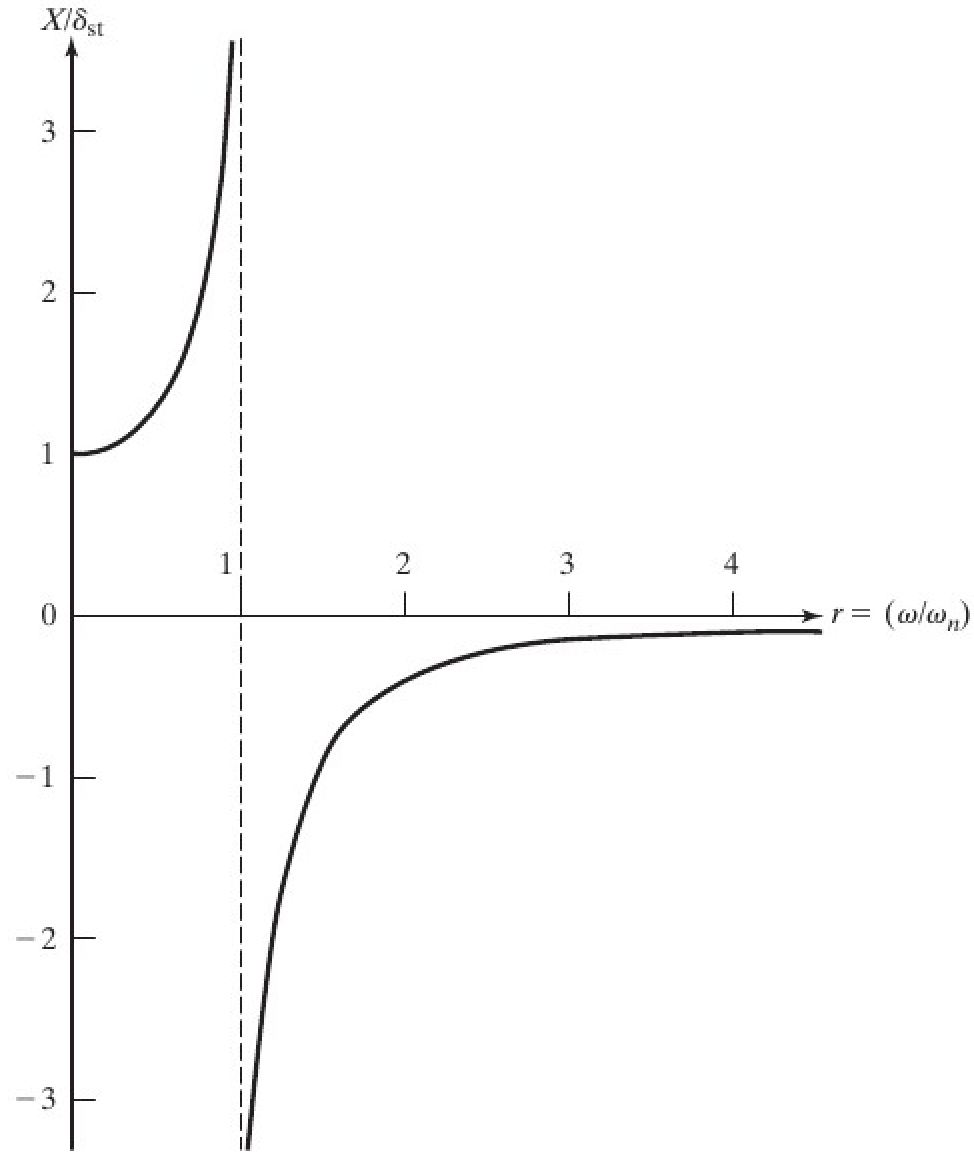
$$\delta_{st} = F_0 / k$$

$$X = \frac{F_0}{k - m\omega^2} = \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

where X is an constant that denotes the maximum amplitude of $x_p(t)$

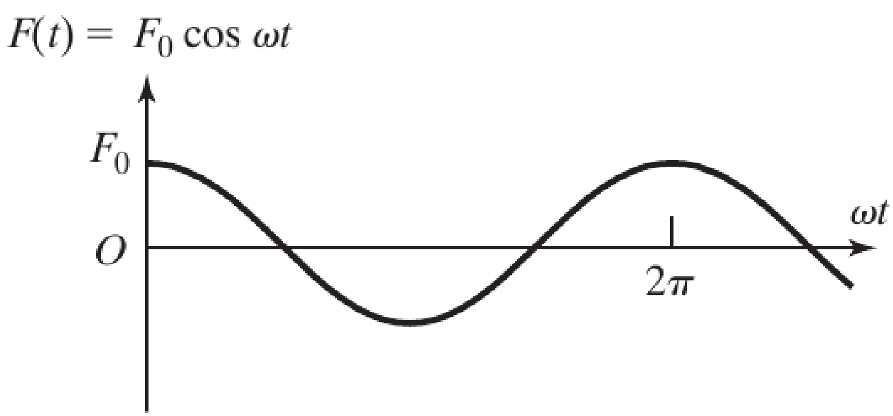
Particular Solution

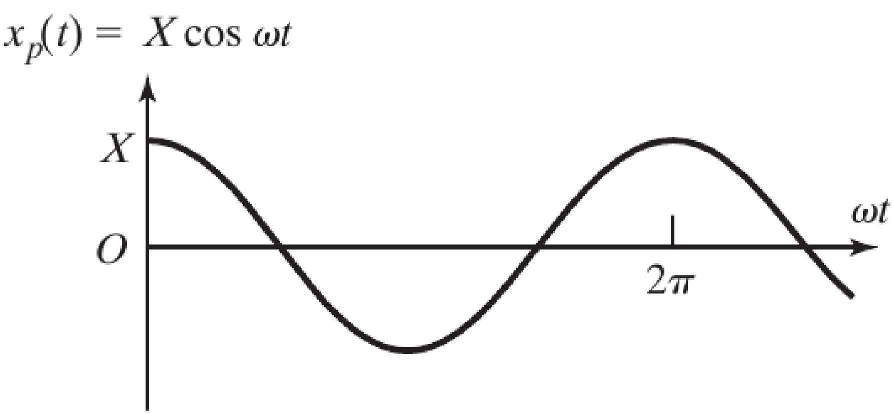
$$X = \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

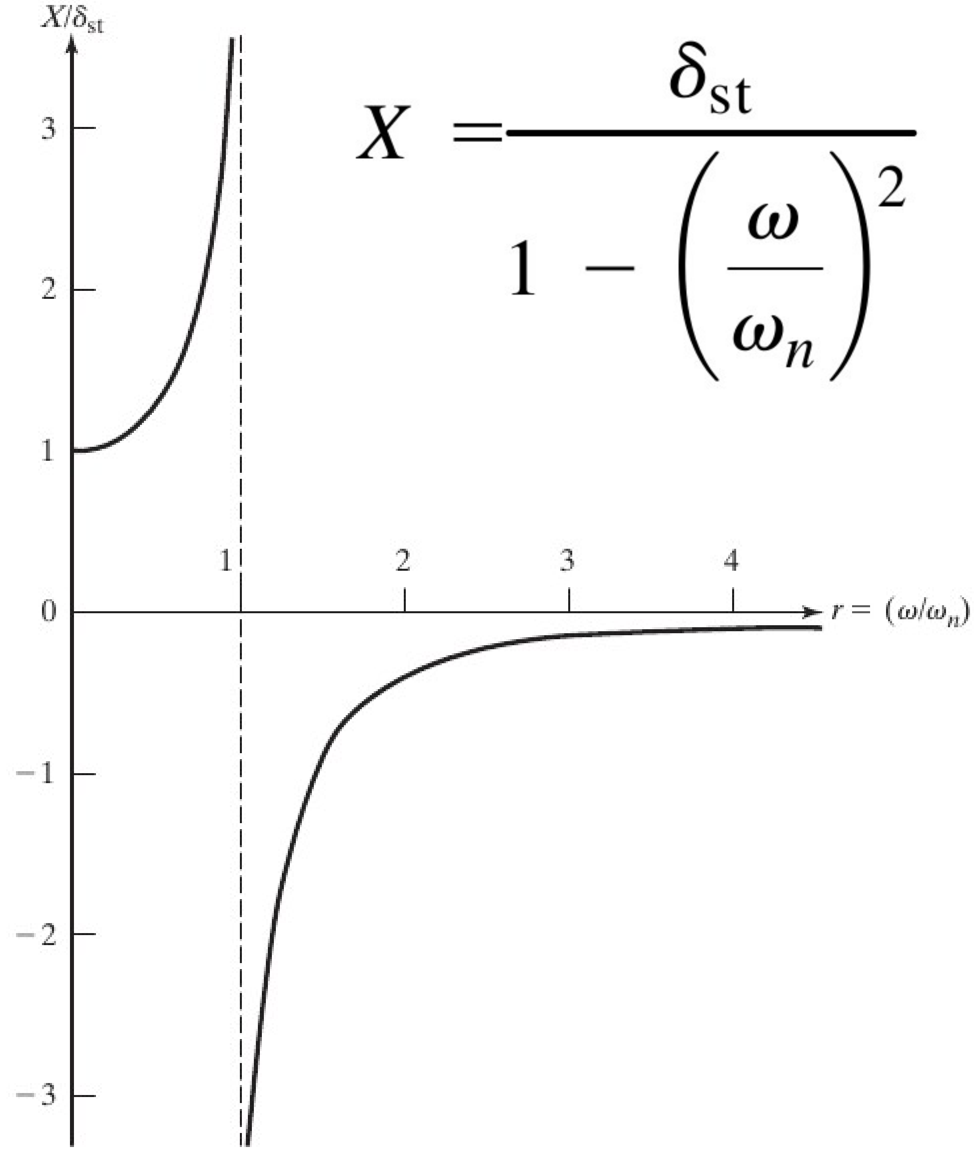


Particular Solution

Case 1. When : $0 < \omega/\omega_n < 1$



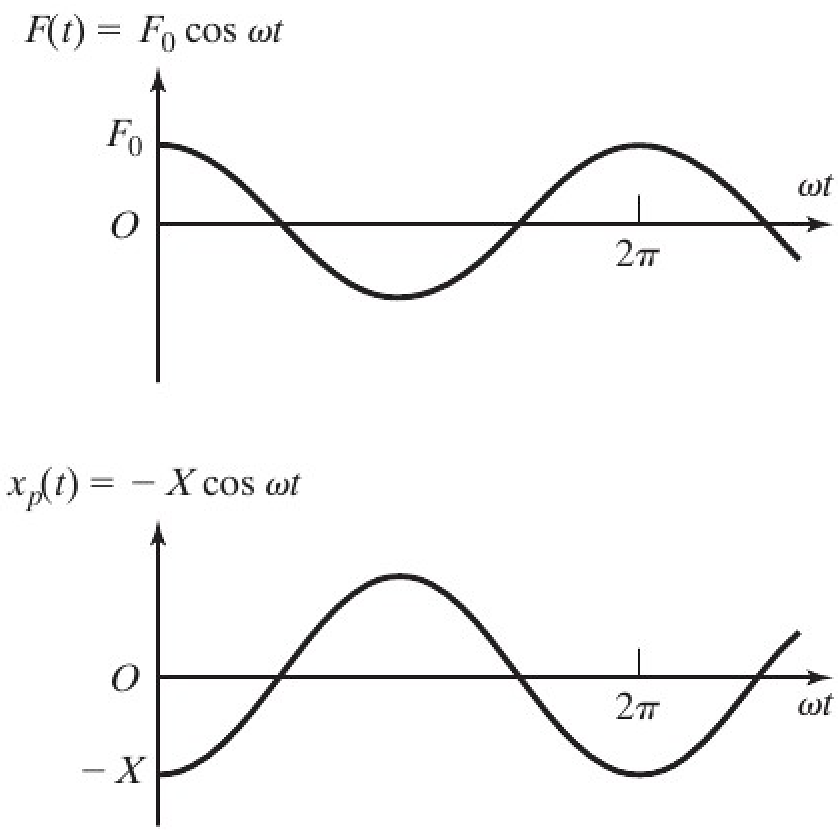




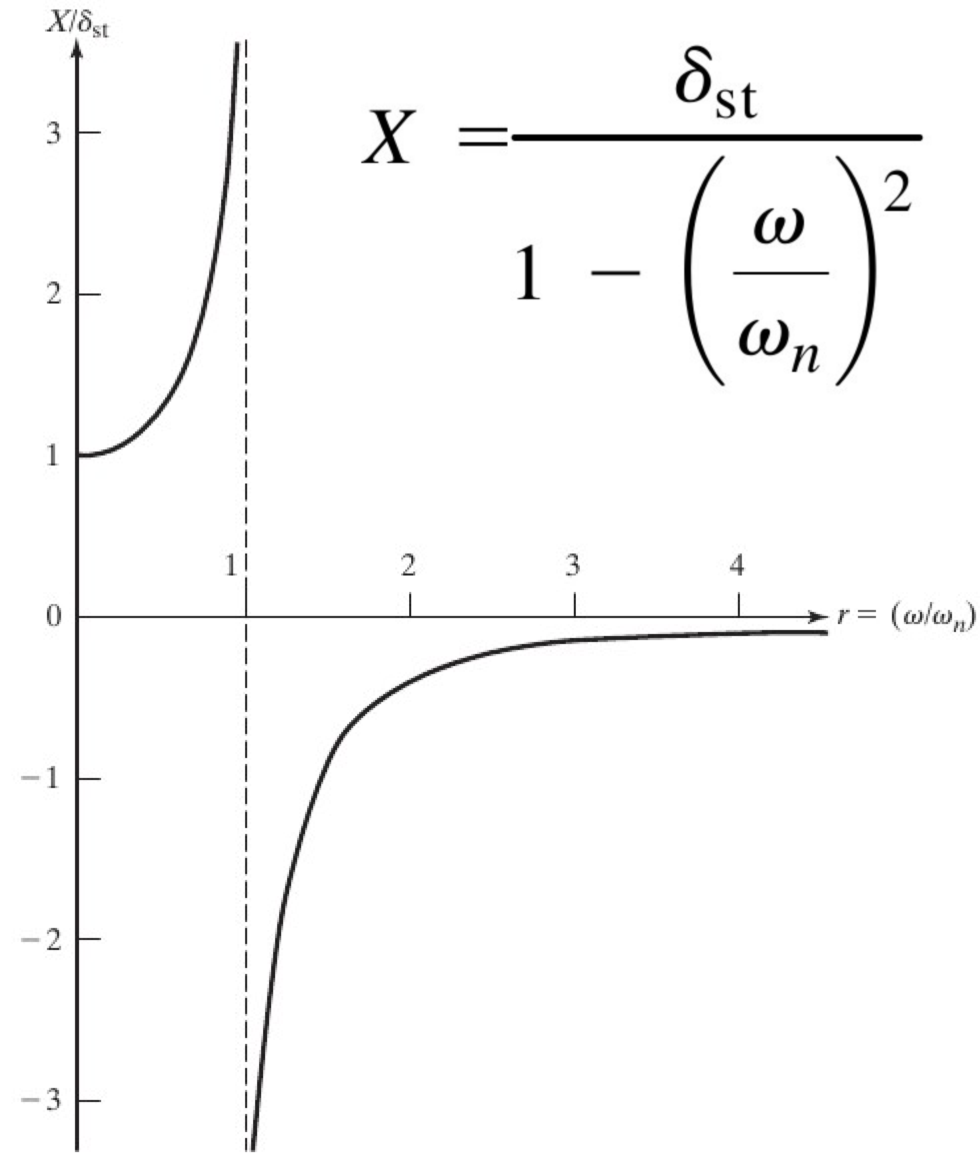
$$X = \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

Particular Solution

Case 2. When : $\omega/\omega_n > 1$



$$X = \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$



General Solution

$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t$$

$$C_1 = x_0 - \frac{F_0}{k - m\omega^2}, \qquad C_2 = \frac{\dot{x}_0}{\omega_n}$$

General Solution

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t$$

$$+ \left(\frac{\dot{x}_0}{\omega_n} \right) \sin \omega_n t + \left(\frac{F_0}{k - m\omega^2} \right) \cos \omega t$$

Particular Solution

Case 3. When : $\omega/\omega_n = 1$

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos\omega_n t + \left(\frac{\dot{x}_0}{\omega_n}\right)\sin\omega_n t + \left(\frac{F_0}{k - m\omega^2}\right)\cos\omega t$$

$$x(t) = x_0 \cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \delta_{st}\left[\frac{\cos\omega t - \cos\omega_n t}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]$$

Particular Solution

Case 3. When : $\omega/\omega_n = 1$

Apply L'hospital's rule

$$\lim_{\omega \to \omega_n}\left[\frac{\cos \omega t - \cos \omega_n t}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2}\right] = \lim_{\omega \to \omega_n}\left[\frac{\dfrac{d}{d\omega}(\cos \omega t - \cos \omega_n t)}{\dfrac{d}{d\omega}\left(1 - \dfrac{\omega^2}{\omega_n^2}\right)}\right]$$

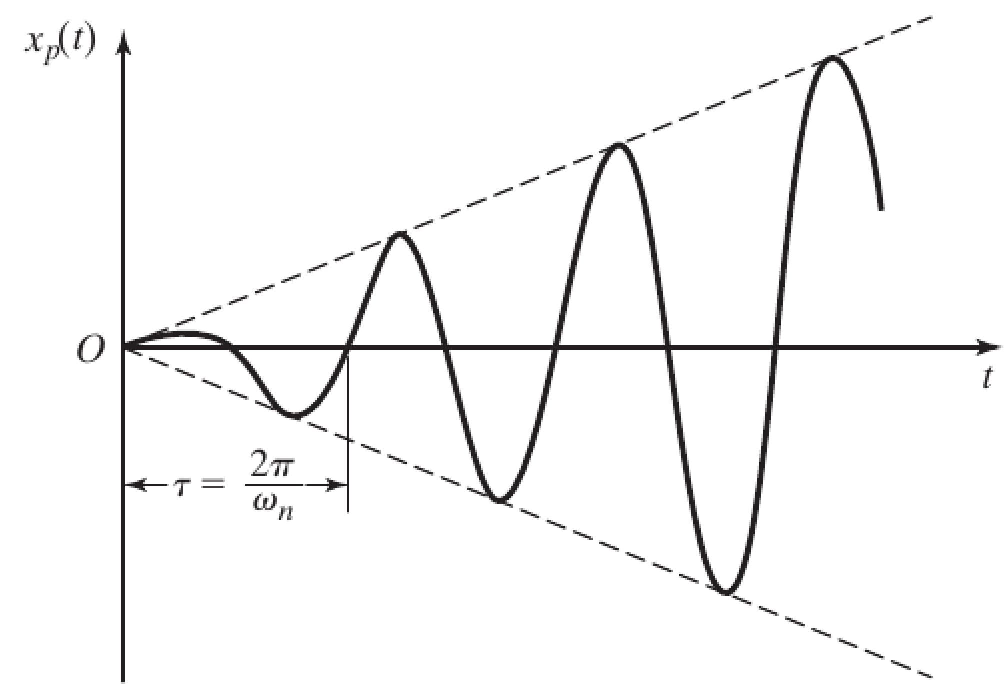
$$= \lim_{\omega \to \omega_n}\left[\frac{t \sin \omega t}{2\,\dfrac{\omega}{\omega_n^2}}\right] = \frac{\omega_n t}{2} \sin \omega_n t$$

Particular Solution

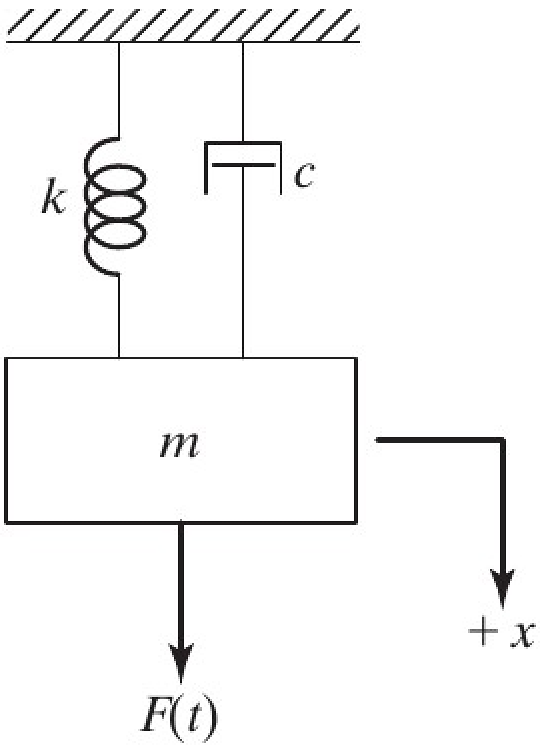
Case 3. When : $\omega/\omega_n = 1$

The response at resonance becomes

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{\delta_{st}\omega_n t}{2} \sin \omega_n t$$



Response of a Damped System Under Harmonic Force



$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

Response of a Damped System Under Harmonic Force

We are interested in the Particular Solution

$$x_p(t) = X \cos(\omega t - \phi)$$

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

$$X[(k - m\omega^2)\cos(\omega t - \phi) - c\omega \sin(\omega t - \phi)] = F_0 \cos \omega t$$

Response of a Damped System Under Harmonic Force

$$X[(k - m\omega^2)\cos(\omega t - \phi) - c\omega\sin(\omega t - \phi)] = F_0\cos\omega t$$

Using

$$\cos(\omega t - \phi) = \cos\omega t\cos\phi + \sin\omega t\sin\phi$$
$$\sin(\omega t - \phi) = \sin\omega t\cos\phi - \cos\omega t\sin\phi$$

$$X[(k - m\omega^2)\cos\phi + c\omega\sin\phi] = F_0$$
$$X[(k - m\omega^2)\sin\phi - c\omega\cos\phi] = 0$$

Response of a Damped System Under Harmonic Force

$$X[(k - m\omega^2)\cos\phi + c\omega\sin\phi] = F_0$$
$$X[(k - m\omega^2)\sin\phi - c\omega\cos\phi] = 0$$

$$X = \frac{F_0}{[(k - m\omega^2)^2 + c^2\omega^2]^{1/2}}$$

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$

Response of a Damped System Under Harmonic Force

$$X = \frac{F_0}{[(k - m\omega^2)^2 + c^2\omega^2]^{1/2}}$$

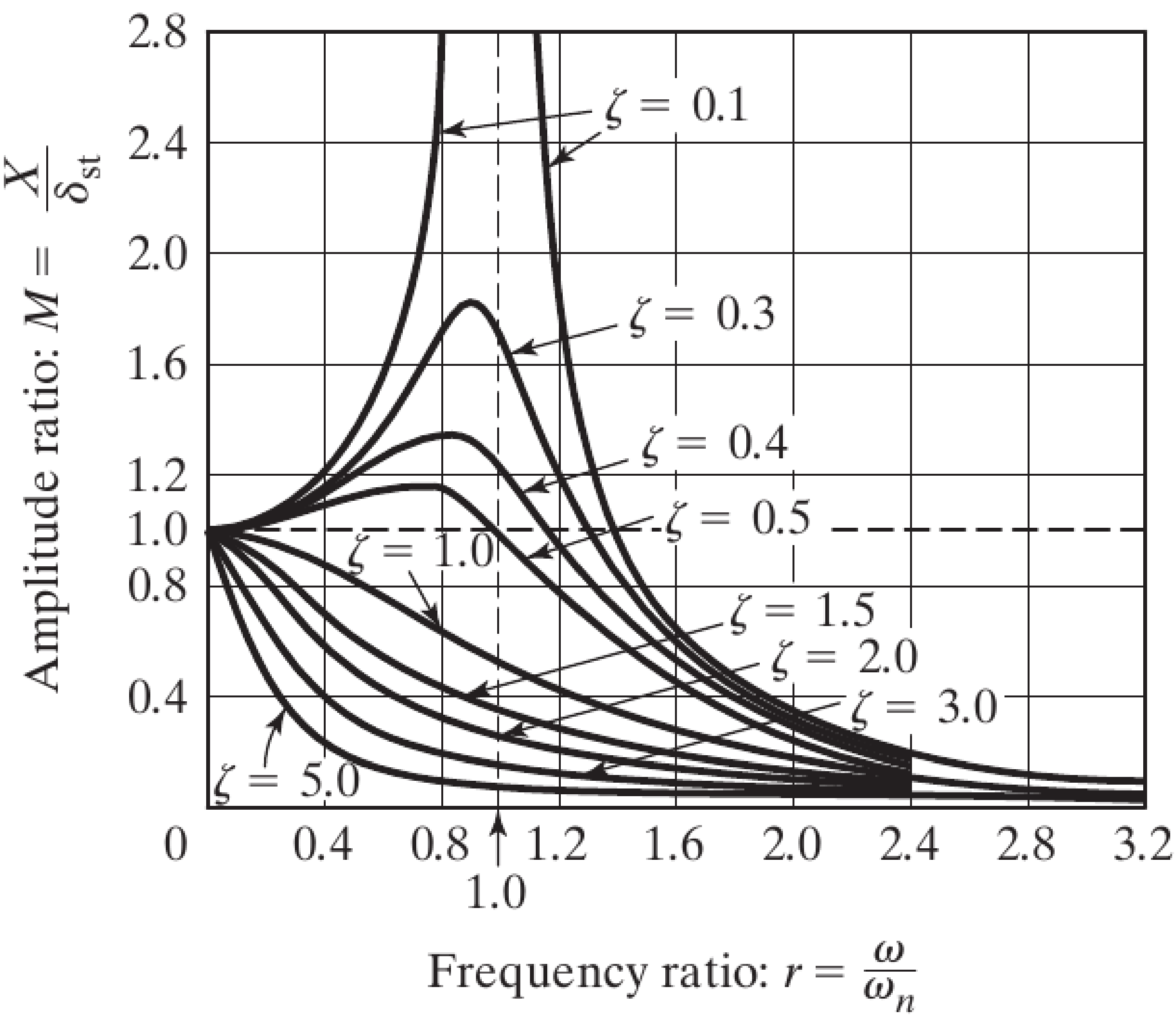
$$\downarrow$$

$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) \rightarrow \phi = \tan^{-1}\left\{\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\} = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$

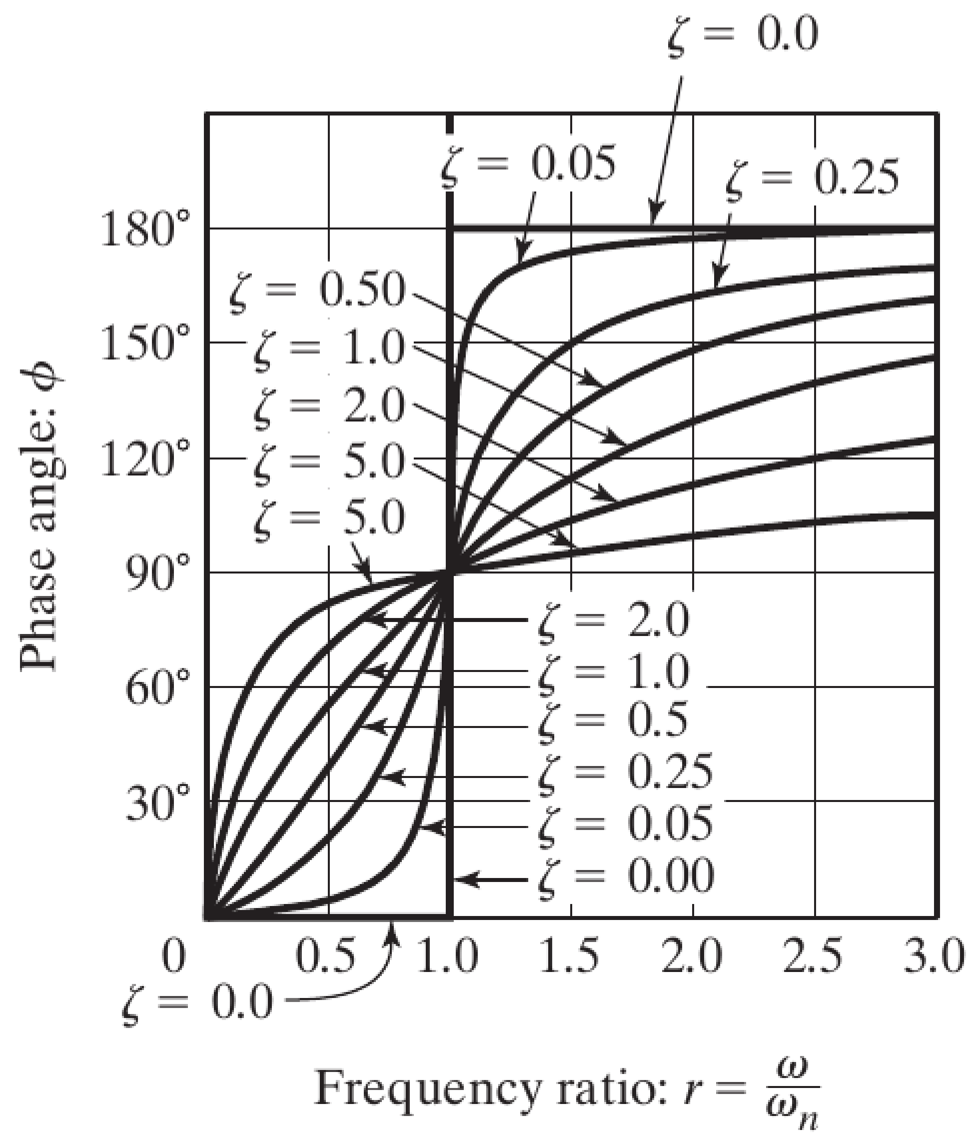
Response of a Damped System Under Harmonic Force

$$\frac{X}{\delta_{st}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

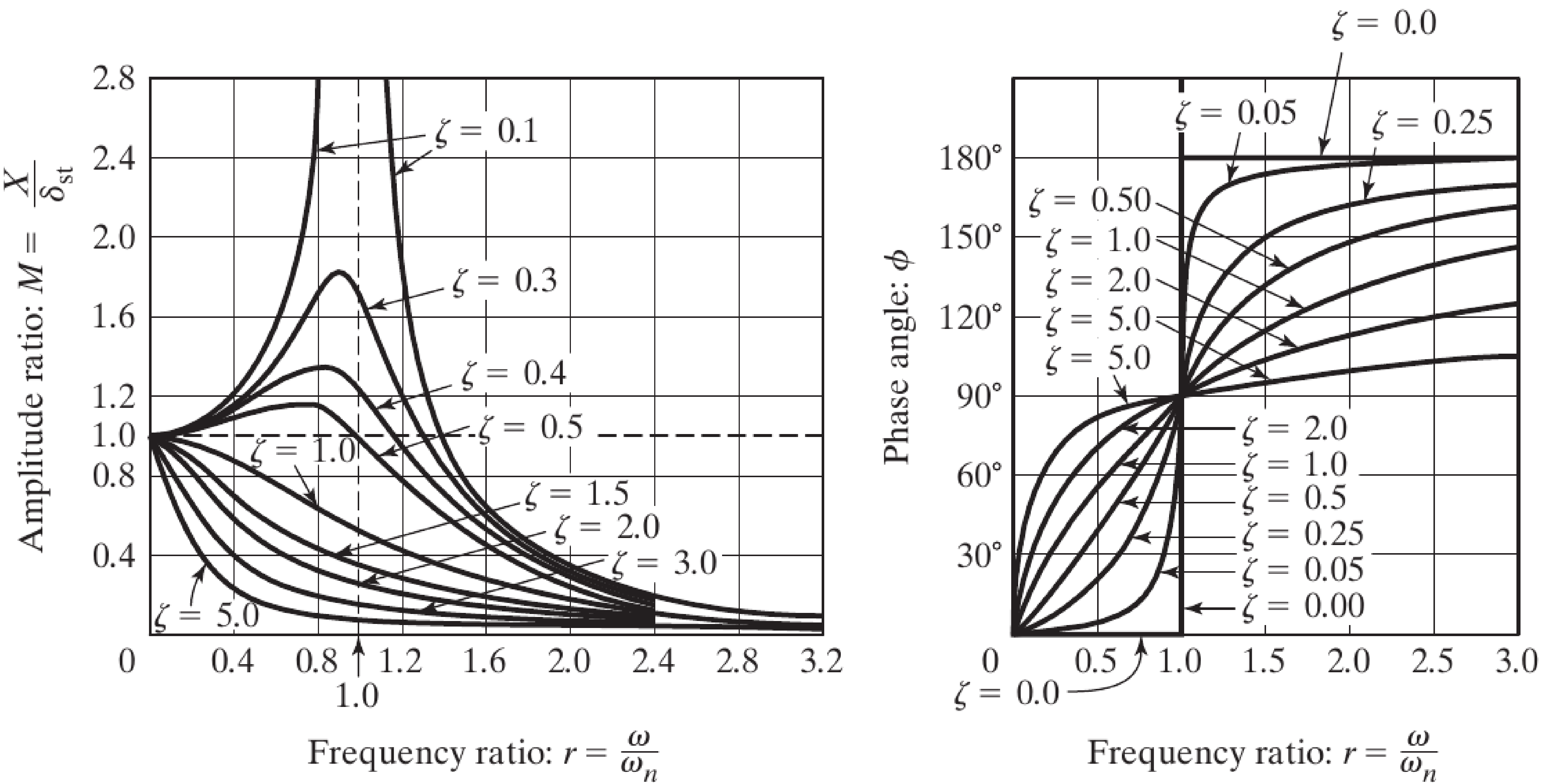


Response of a Damped System Under Harmonic Force

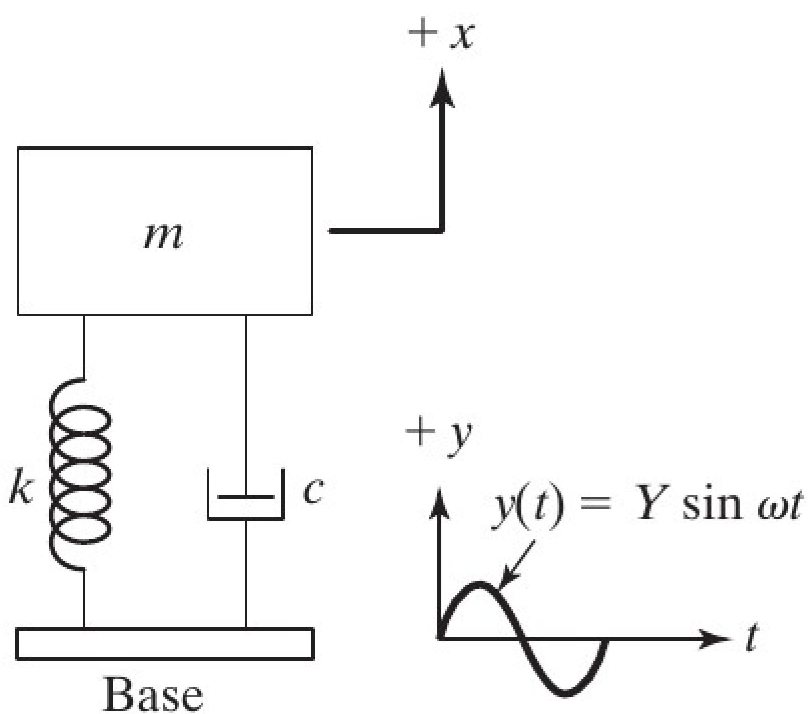
$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$

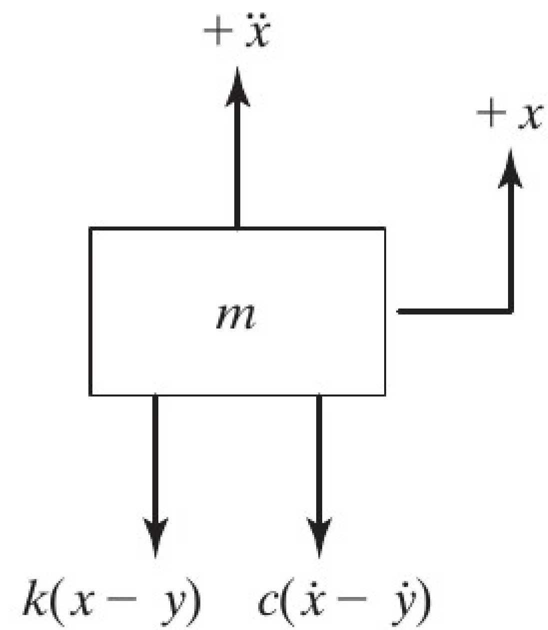


Response of a Damped System Under Harmonic Force

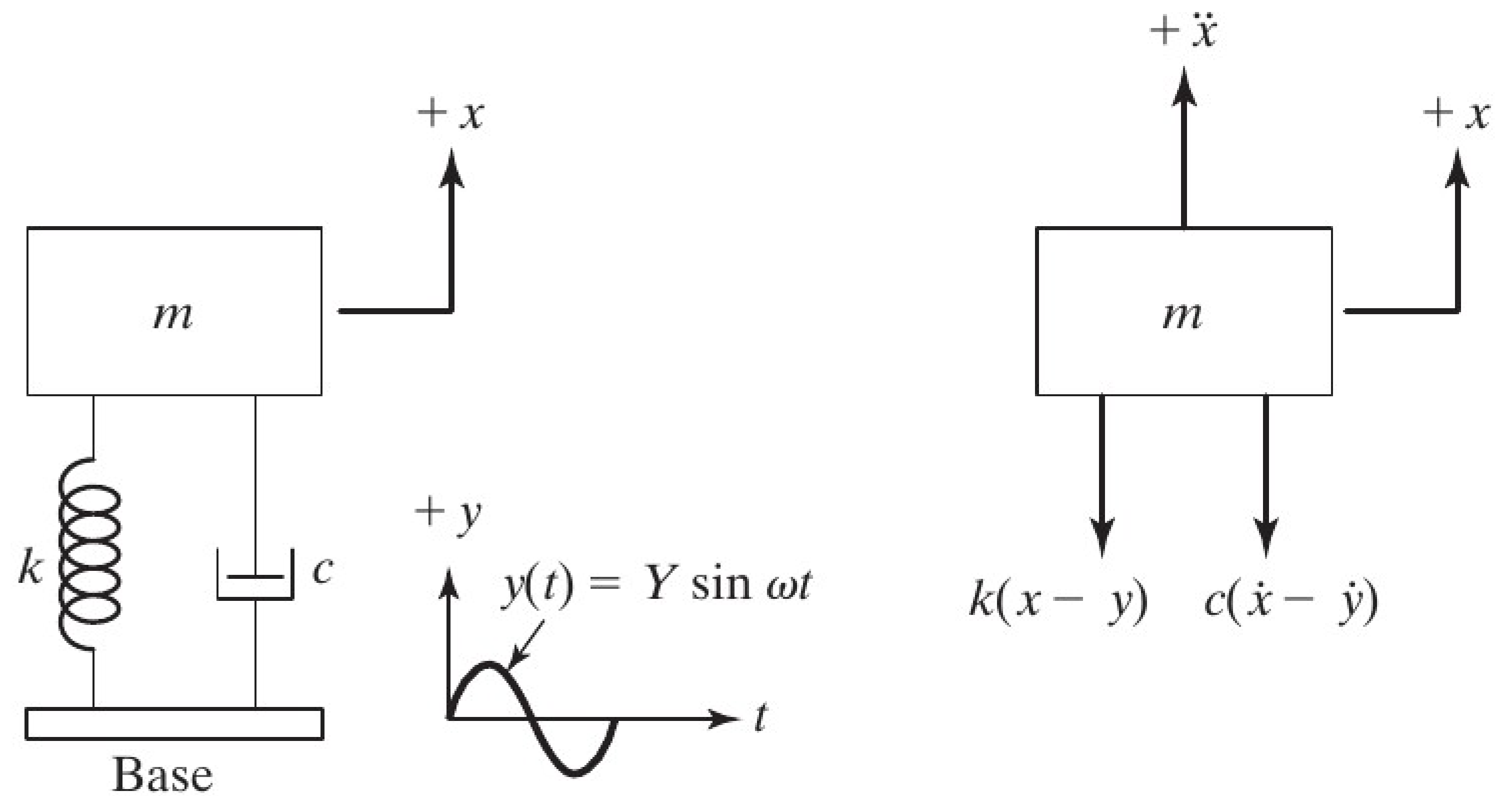


Response of a Damped System Under the Harmonic Motion of the Base





Response of a Damped System Under the Harmonic Motion of the Base



$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

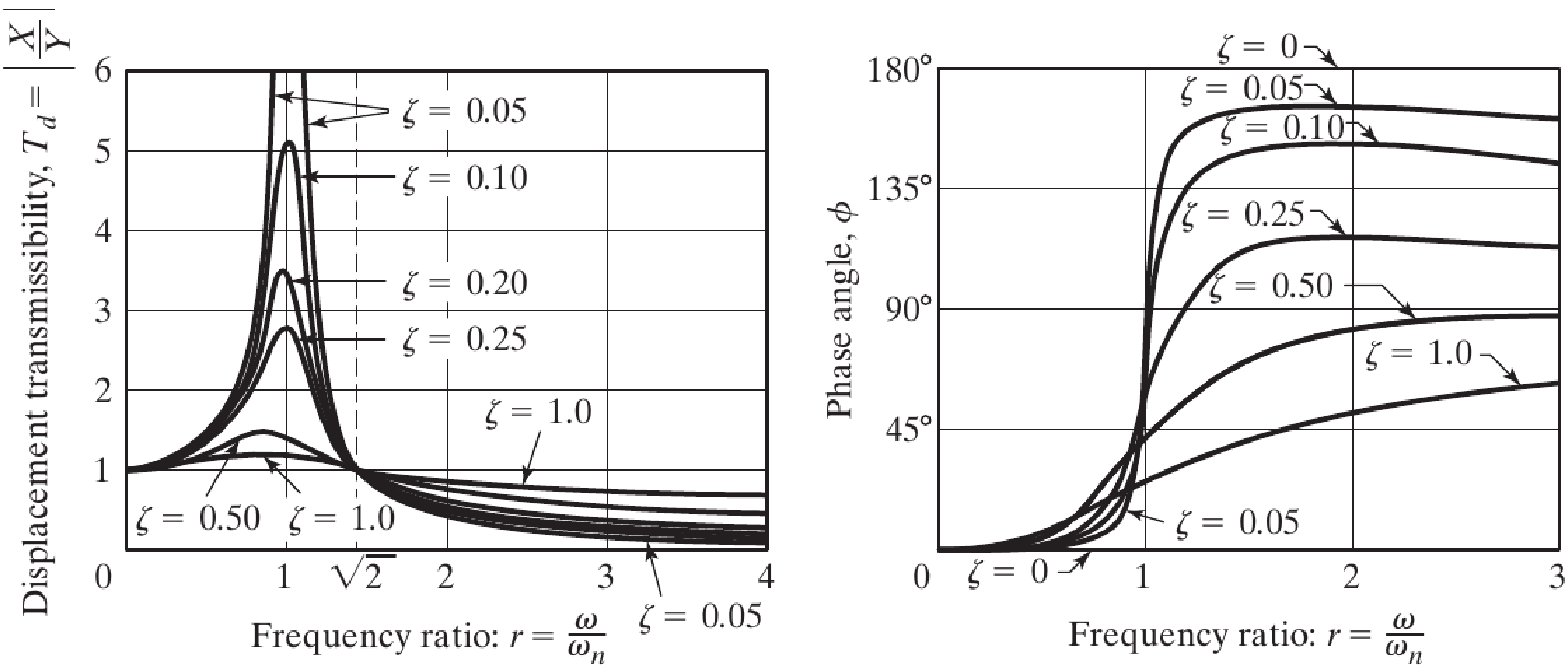
Displacement Transmissibility

$$\frac{X}{Y} = \left[\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}\right]^{1/2} = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

$$\phi = \tan^{-1}\left[\frac{mc\omega^3}{k(k - m\omega^2) + (\omega c)^2}\right] = \tan^{-1}\left[\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\right]$$

Transmissibility

$$\frac{X}{Y} = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2}$$



Transmitted Force

The transmitted to the base or support due to the reactions from the spring and the dashpot

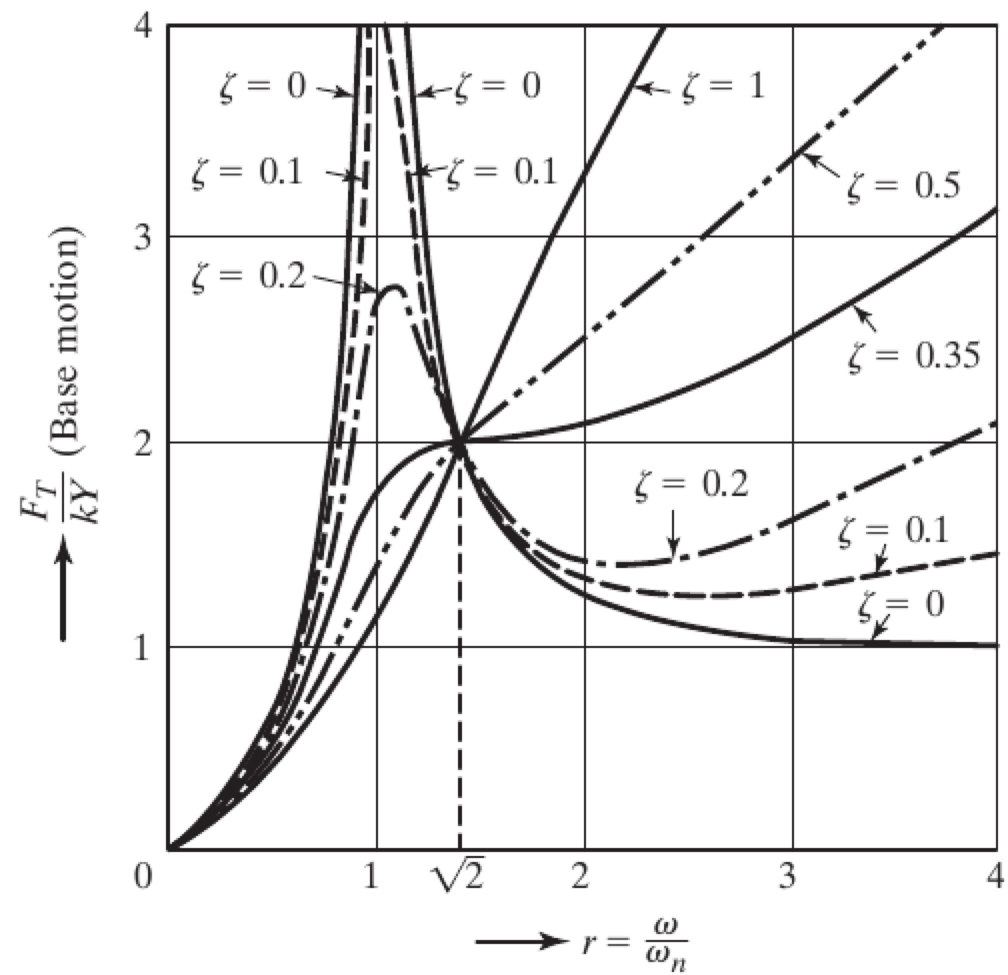
$$F = k(x - y) + c(\dot{x} - \dot{y}) = -m\ddot{x}$$

$$F = m\omega^2 X \sin(\omega t - \phi) = F_T \sin(\omega t - \phi)$$

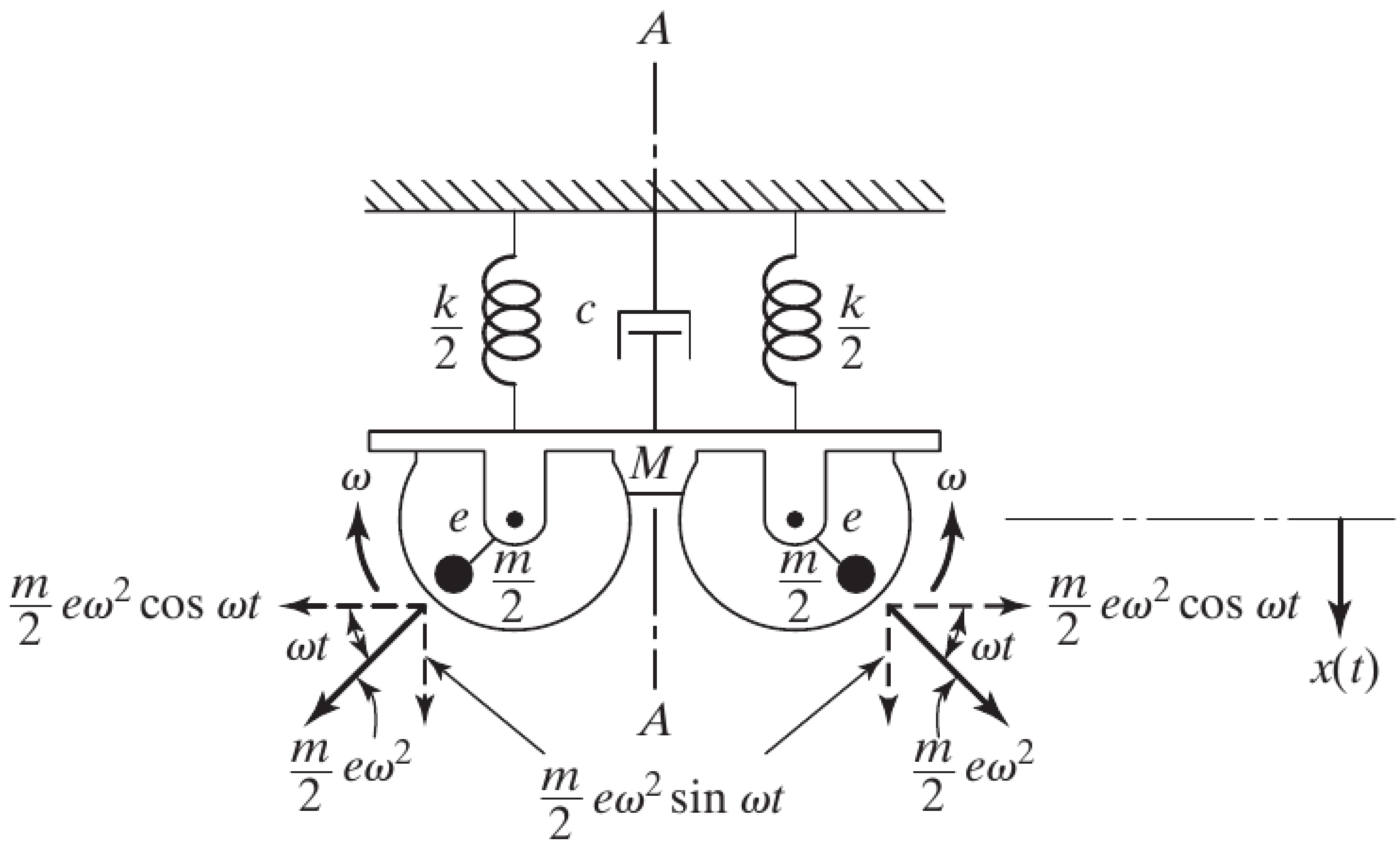
$$\frac{F_T}{kY} = r^2 \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2}$$

Transmitted Force

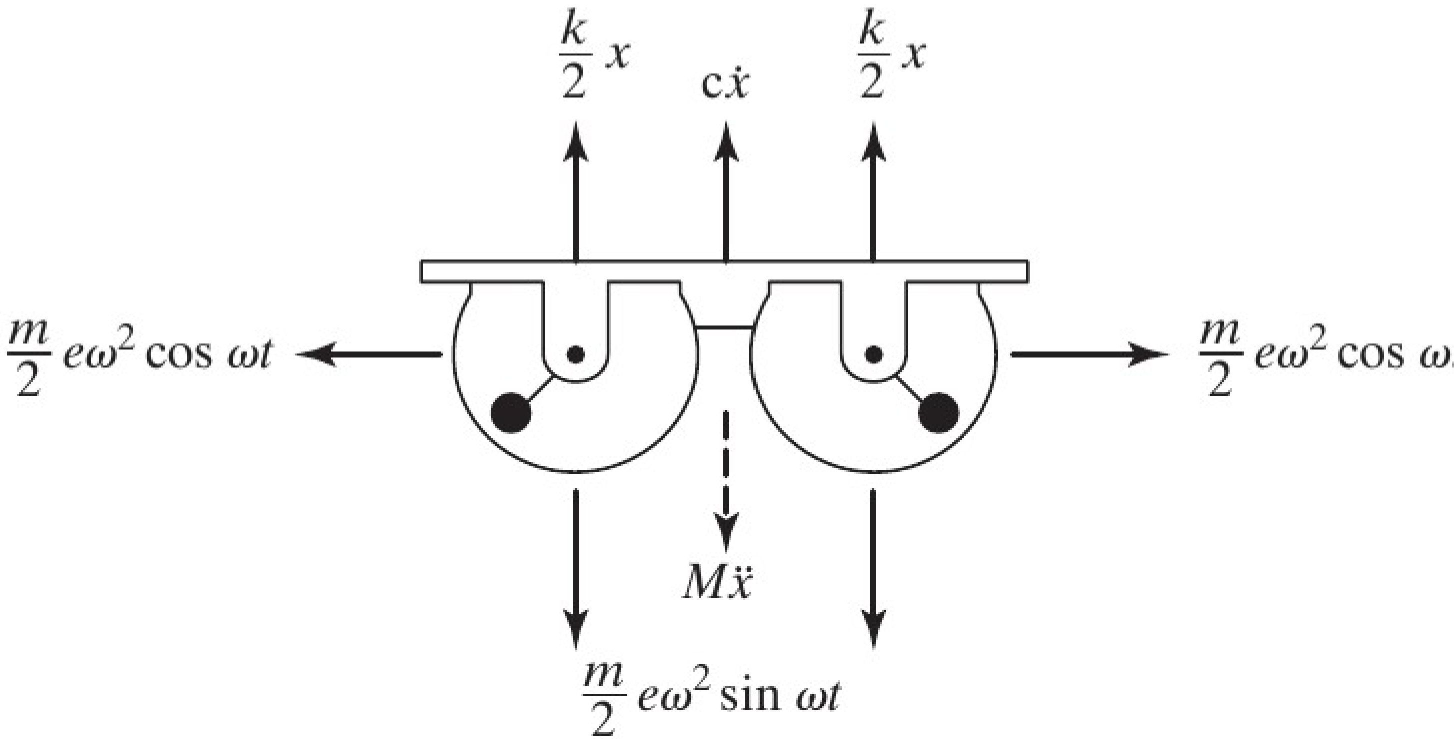
$$\frac{F_T}{kY} = r^2\left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$



Response of a Damped System Under Rotating Unbalance



Response of a Damped System Under Rotating Unbalance



$$M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin\omega t$$

$$X = \frac{me\omega^2}{\left[(k - M\omega^2)^2 + (c\omega)^2\right]^{1/2}}$$

Response of a Damped System Under Rotating Unbalance

$$\frac{MX}{me} = \frac{r^2}{\left[(1 - r^2)^2 + (2\zeta r)^2\right]^{1/2}}$$